

# MATH 3969 - MEASURE THEORY AND FOURIER ANALYSIS

ANDREW TULLOCH

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1. MEASURE THEORY

**Definition 1.1** ( $\sigma$ -algebra). Let  $X$  be a set. A collection  $\mathcal{A}$  of subsets of  $X$  is called a  $\sigma$ -algebra if

- $\emptyset \in \mathcal{A}$
- If  $A \subseteq X$  is in  $\mathcal{A}$ , then its complement  $A^c = X \setminus A$  is in  $\mathcal{A}$
- Whenever  $A_0, A_1, \dots$  are subsets of  $X$  in  $\mathcal{A}$ , then their union

$$\bigcup_{k=0}^{\infty} A_k$$

also belongs to  $\mathcal{A}$ .

**Definition 1.2** (Measure). Let  $X$  be a set, and let  $\mathcal{A}$  be a  $\sigma$ -algebra of subsets of  $X$ . Suppose that  $\mu : \mathcal{A} \rightarrow [0, \infty]$  is a function. Then  $\mu$  is a measure if

- $\mu(\emptyset) = 0$
- Whenever  $A_0, A_1, \dots$  are *pairwise disjoint* subsets of  $X$  in  $\mathcal{A}$ , then

$$\mu\left(\bigcup_{k=0}^{\infty} A_k\right) = \sum_{k=0}^{\infty} \mu(A_k)$$

**Proposition 1.3** (Properties of a  $\sigma$ -algebra ). *Let  $\mathcal{A}$  be a  $\sigma$ -algebra of subsets of a set  $X$ . Then*

- $X, \emptyset \in \mathcal{A}$
- If  $A_k \in \mathcal{A}$ , then  $\bigcap_{k=0}^{\infty} A_k \in \mathcal{A}$
- If  $A, B \in \mathcal{A}$ , then  $A \cup B, A \cap B \in \mathcal{A}$

**Definition 1.4** (Algebra). A collection  $\mathcal{A}$  of subsets  $A$  of  $X$  which satisfies the first two conditions of a  $\sigma$ -algebra and also

- If  $A, B \in \mathcal{A}$ , then  $A \cup B \in \mathcal{A}$

is called an algebra. Every  $\sigma$ -algebra is an algebra, but not every algebra is a  $\sigma$ -algebra

**Definition 1.5** ( $\sigma$ -algebra generated by  $\mathcal{S}$ ). Let  $\mathcal{S}$  be a collection of subsets of  $X$ . Let

$$\mathcal{A}(\mathcal{S}) = \bigcap \{ \mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-algebra, and } \mathcal{S} \subseteq \mathcal{A} \}$$

$\mathcal{A}(\mathcal{S})$  is called the  $\sigma$ -algebra generated by  $\mathcal{S}$

**Definition 1.6** (Borel  $\sigma$ -algebra ). Let  $X$  be a metric space and  $\mathcal{S}$  the collection of all open sets in  $X$ . We call  $\mathcal{B} = \mathcal{A}(\mathcal{S})$  the *Borel  $\sigma$ -algebra* . Sets in  $\mathcal{B}$  are called *Borel sets*.

**Corollary.** *We have the following examples of Borel sets.*

- Any open set is a Borel set.
- If  $B$  is a Borel set, then so is  $B^c$ . If  $B_0, B_1, \dots$  is a sequence of Borel sets, then so are  $\bigcup_{k=0}^{\infty} B_k$  and  $\bigcap_{k=0}^{\infty} B_k$ .

1.1. Properties of Measures.

**Proposition 1.7** (The Monotonicity Property). *If  $A$  and  $B$  are  $\mu$ -measurable subsets of  $X$  with  $A \subseteq B$ , then  $\mu(A) \leq \mu(B)$ .*

**Proposition 1.8** (The Countable Subadditivity Property). *If  $A_0, A_1, \dots$  are  $\mu$ -measurable subsets of  $X$ , then*

$$\mu\left(\bigcup_{k=0}^{\infty} A_k\right) \leq \sum_{k=0}^{\infty} \mu(A_k)$$

**Proposition 1.9** (Monotone Convergence Property of Measures). *Let  $A_0 \subseteq A_1 \subseteq A_2$  be an increasing sequence of measurable sets. Then*

$$\mu\left(\bigcup_{k=0}^{\infty} A_k\right) = \lim_{k \rightarrow \infty} \mu(A_k)$$

**Proposition 1.10.** *Let  $A_0 \supseteq A_1 \supseteq A_2 \dots$  be sets from some  $\sigma$ -algebra  $\mathcal{A}$ . If  $\mu(A_0) < \infty$ , then*

$$\mu\left(\bigcap_{k=0}^{\infty} A_k\right) = \lim_{k \rightarrow \infty} \mu(A_k)$$

1.2. Constructing  $\sigma$ -algebras and measures.

**Definition 1.11** (Lebesgue outer measure). If  $A \subseteq \mathbb{R}$ , let

$$m^*(A) = \inf \left\{ \sum_{k=0}^{\infty} (b_k - a_k) \mid a_k < b_k, A \subseteq \bigcup_{k=0}^{\infty} (a_k, b_k) \right\}$$

**Proposition 1.12** (Properties of the Lebesgue outer measure). *The Lebesgue measure obeys the following properties.*

- $m^*(A)$  is defined, and  $m^*(A) \in [0, \infty]$  for any subset of  $\mathbb{R}$ .
- $m^*(\emptyset) = 0$
- If  $A \subseteq B$ ,  $m^*(A) \leq m^*(B)$
- For every sequence  $A_0, A_1, \dots$ , we have

$$m^*\left(\bigcup_{k=0}^{\infty} A_k\right) \leq \sum_{k=0}^{\infty} m^*(A_k)$$

**Definition 1.13** (Outer Measure). A function  $\mu^* : \mathcal{P} \rightarrow [0, \infty]$  is such that

- $\mu^*(\emptyset) = 0$
- If  $A \subseteq B$ ,  $\mu^*(A) \leq \mu^*(B)$
- For every sequence  $A_0, A_1, \dots$ , we have

$$\mu^*\left(\bigcup_{k=0}^{\infty} A_k\right) \leq \sum_{k=0}^{\infty} \mu^*(A_k)$$

Then  $\mu^*$  is called an *outer measure* on  $X$ .

**Theorem 1.14** (Construction from outer measures). *Let  $\mu^*$  be an outer measure on a set  $X$ . Then*

$$\mathcal{A} = \{A \subseteq X \mid \mu^*(S) = \mu^*(S \cap A) + \mu^*(S \cap A^c) \text{ for all } S \subseteq X\}$$

*Then  $\mathcal{A}$  is a  $\sigma$ -algebra. Let  $\mu(A) = \mu^*(A)$  when  $A \in \mathcal{A}$ . Then  $\mu : \mathcal{A} \rightarrow [0, \infty]$  is a measure.*

**Proposition 1.15.** *Let  $\mu^*$  be an outer measure, and let  $\mathcal{A}$  be the  $\sigma$ -algebra defined in the last theorem. Let  $A \subseteq X$  satisfy  $\mu^*(A) = 0$ . Then  $A \in \mathcal{A}$ , and so  $\mu(A)$  is defined, and equals 0.*

**Definition 1.16** (Null set). If  $\mu : \mathcal{A} \rightarrow [0, \infty]$  is any measure, then a set  $A \in \mathcal{A}$  satisfying  $\mu(A) = 0$  is called a null set.

**1.3. Properties of the Lebesgue measure on  $\mathbb{R}$ .** Let

$$\mathcal{M} = \{A \subseteq \mathbb{R} \mid m^*(S) = m^*(S \cap A) + m^*(S \cap A^c) \text{ for all } S \subseteq \mathbb{R}\}$$

The sets in  $\mathcal{M}$  are called the *Lebesgue measurable subsets* of  $\mathbb{R}$ . If  $A \in \mathcal{M}$ , then we write  $m(A) = m^*(A)$ . This real number is called the *Lebesgue measure* of  $A$ .

We now show that this  $\sigma$ -algebra  $\mathcal{M}$  is very large.

**Theorem 1.17.** *Let  $m^*$  denote the Lebesgue outer measure on  $\mathbb{R}$ , and let  $\mathcal{M}$  be the  $\sigma$ -algebra of Lebesgue measurable sets. Then*

- *If  $I \subseteq \mathbb{R}$  is an interval. Then  $m^*(I) = l(I)$ . That is, the outer measure is just its length.*
- *If  $I \subseteq \mathbb{R}$  is an interval, then  $I \in \mathcal{M}$ .*

**Proposition 1.18.** *Any open subset of  $\mathbb{R}$  is in  $\mathcal{M}$ . Any closed subset of  $\mathbb{R}$  is in  $\mathcal{M}$ . That is, all open or closed sets in  $\mathbb{R}$  are Lebesgue measurable.*

**Corollary.** *Every Borel subset of  $\mathbb{R}$  is contained in  $\mathcal{M}$ .*

*Proof.*  $\mathcal{M}$  is a  $\sigma$ -algebra which contains every open subset of  $\mathbb{R}$ . The  $\sigma$ -algebra  $\mathcal{B}$  is by definition the smallest such  $\sigma$ -algebra. Thus  $\mathcal{B} \subseteq \mathcal{M}$ . □

2. MEASURABLE FUNCTIONS

**Definition 2.1** (Measurable function). Let  $\mathcal{A}$  be  $\sigma$ -algebra of subsets of a set  $X$ . A function  $f : X \rightarrow \overline{\mathbb{R}}$ , is called *measurable* (or  $\mathcal{A}$ -measurable) if for every  $\alpha \in \mathbb{R}$ , the set

$$\{x \in X \mid f(x) > \alpha\}$$

is in  $\mathcal{A}$ .

**Definition 2.2** (Indicator function). Let  $S \subset X$ . We define the *indicator function* of  $S$  to be the function  $1_S : X \rightarrow \mathbb{R}$  given by

$$1_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

**Proposition 2.3.** *Let  $S \subset X$ . Then  $1_S$  is measurable if and only if  $S \in \mathcal{A}$ .*

*Proof.* Let  $\alpha \in \mathbb{R}$ . Then

$$\{x \in X \mid 1_S(x) > \alpha\} = \begin{cases} \emptyset & \text{if } \alpha \geq 1 \\ S & \text{if } 0 \leq \alpha < 1 \\ X & \text{if } \alpha < 0 \end{cases}$$

As  $\emptyset, X$  are in  $\mathcal{A}$ , then  $1_S$  is measurable if and only if  $S \in \mathcal{A}$ . □

**Proposition 2.4** (Continuous functions are Lebesgue measurable). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. Then  $f$  is Lebesgue measurable on  $[a, b]$ . More generally, if  $X \subset \mathbb{R}$  is in  $\mathcal{M}$  and  $f : X \rightarrow \mathbb{R}$  is continuous, then  $f$  is Lebesgue measurable on  $X$ .*

### 2.1. Basic properties of measurable functions.

**Lemma 2.5.** *Let  $\mathcal{A}$  be a  $\sigma$ -algebra of subsets of a set  $X$ , and let  $f : X \rightarrow \overline{\mathbb{R}}$  be a function. Then  $f$  is measurable if and only if it satisfies one of the following conditions.*

- For each  $\alpha \in \mathbb{R}$ , the set  $\{x \in X \mid f(x) > \alpha\}$  is in  $\mathcal{A}$ .
- For each  $\alpha \in \mathbb{R}$ , the set  $\{x \in X \mid f(x) < \alpha\}$  is in  $\mathcal{A}$ .
- For each  $\alpha \in \mathbb{R}$ , the set  $\{x \in X \mid f(x) \leq \alpha\}$  is in  $\mathcal{A}$ .
- For each  $\alpha \in \mathbb{R}$ , the set  $\{x \in X \mid f(x) \geq \alpha\}$  is in  $\mathcal{A}$ .

**Proposition 2.6.** *Let  $\mathcal{A}$  be a  $\sigma$ -algebra of subsets of a set  $X$ , and let  $f, g : X \rightarrow \overline{\mathbb{R}}$  be functions. Then*

- $f + g$  is measurable (provided that  $f(x) = \infty$  and  $g(x) = -\infty$  or vice versa holds for no  $x \in X$ ).
- $cf$  is measurable for any constant  $c \in \mathbb{R}$ .
- $fg$  is measurable.
- $f/g$  is measurable (provided that  $g(x)$  is nonzero and not infinity for all  $x \in X$ ).

Similarly, let  $f_0, f_1, \dots : X \rightarrow \overline{\mathbb{R}}$ . Then

- $\sup\{f_0, f_1, \dots\}$  and  $\inf\{f_0, f_1, \dots\}$  are measurable functions.

**Corollary.** *Let  $f, g$  be measurable. Then  $\max\{f, g\}$  and  $\min\{f, g\}$  are measurable functions.*

**Proposition 2.7.** *Let  $\mathcal{A}$  be a  $\sigma$ -algebra of subsets of a set  $X$ , and let  $f_0, f_1, \dots : X \rightarrow \overline{\mathbb{R}}$  be measurable functions. Let  $f(x) = \lim_{k \rightarrow \infty} f_k(x)$  for each  $x \in X$ . Then  $f$  is a measurable function.*

2.2. Simple functions.

**Definition 2.8** (Simple function). Let  $\mathcal{A}$  be a  $\sigma$ -algebra of subsets of a set  $X$ . A function  $\varphi : X \rightarrow \mathbb{R}$  is called *simple* if it is measurable and only takes a finite number of values.

**Proposition 2.9.** Let  $\mathcal{A}$  be a  $\sigma$ -algebra of subsets of a set  $X$ , and let  $f : X \rightarrow [0, \infty]$  be a nonnegative measurable function. Then there is a sequence  $(\varphi_n)$  of simple functions such that

- $0 \leq \varphi_1(x) \leq \varphi_2(x) \leq \dots \leq f(x)$  for all  $x \in X$ .
- $f(x) = \lim_{n \rightarrow \infty} \varphi_n(x)$  for all  $x \in X$ .

*Proof.* Define the function  $\varphi_n$  as follows.

Let

$$A_{n,k} = \{x \in X \mid \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}\}$$

and let  $A_{n,2^n} = \{x \in X \mid f(x) \geq n\}$

Then the function

$$\varphi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} 1_{A_{n,k}}$$

obeys the required properties. □

3. INTEGRATION

**Definition 3.1** (Integration of simple functions). Let  $\varphi = \sum_{j=1}^m a_j 1_{A_j}$ . Then the *integral*  $\int_X \varphi d\mu$  of  $\varphi$  over  $X$  with respect to  $\mu$  is given by

$$\int_X \varphi d\mu = \sum_{j=1}^m a_j \mu(A_j)$$

**Proposition 3.2.** Let  $\varphi$  and  $\psi$  be nonnegative simple functions on  $X$ , and let  $c \geq 0$  be constant. Then

- $\int_X \varphi + \psi d\mu = \int_X \varphi d\mu + \int_X \psi d\mu$
- If  $0 \leq \psi \leq \varphi$ , then

$$0 \leq \int_X \psi d\mu \leq \int_X \varphi d\mu$$

- $\int_X c\varphi d\mu = c \int_X \varphi d\mu$

**Definition 3.3** (Integral over a subset of  $X$ ). Let  $\varphi$  be a nonnegative simple function, and let  $S \subset X$  be measurable. Then the *integral of  $\varphi$  over  $S$  with respect to  $\mu$* , denoted  $\int_S \varphi d\mu$ , is given by

$$\int_S \varphi d\mu = \int_X \varphi \cdot 1_S d\mu$$

**3.1. Integration of nonnegative measurable functions.**

**Definition 3.4.** Let  $f : X \rightarrow [0, \infty]$  be a nonnegative measurable function. We define the *integral*  $\int_X f d\mu$  of  $f$  over  $X$  with respect to  $\mu$  by

$$\int_X f d\mu = \sup \left\{ \int_X \varphi d\mu \mid \varphi \text{ is simple, and } 0 \leq \varphi \leq f \text{ on } X \right\}$$

**Lemma 3.5.** Suppose that  $f, g$  are two nonnegative measurable functions, and  $0 \leq g \leq f$  on  $X$ . Then

$$0 \leq \int_X g d\mu \leq \int_X f d\mu$$

The following is an extremely important theorem in measure theory.

**Theorem 3.6** (Monotone convergence theorem). Let  $(f_k)$  be a sequence of nonnegative measurable functions on  $X$ . Assume that

- $0 \leq f_0(x) \leq f_1(x) \leq \dots$  for each  $x \in X$ ,
- $\lim_{k \rightarrow \infty} f_k(x) = f(x)$  for each  $x \in X$ .

When these hold, we write  $f_k \nearrow f$  pointwise.

Then  $f$  is measurable, and

$$\int_X f d\mu = \lim_{k \rightarrow \infty} \int_X f_k d\mu$$

**Corollary.** Let  $f, g$  be measurable on  $X$ . Then

$$\int_X f + g d\mu = \int_X f d\mu + \int_X g d\mu$$

**Theorem 3.7.** Suppose that  $f_k$  is a nonnegative measurable function for  $k = 0, 1, \dots$ . Then

$$\int_X \left( \sum_{k=0}^{\infty} f_k \right) d\mu = \sum_{k=0}^{\infty} \left( \int_X f_k d\mu \right)$$

**Theorem 3.8.** Suppose that  $X$  is a set,  $\mathcal{A}$  is a  $\sigma$ -algebra of subsets of  $X$  and  $\mu : \mathcal{A} \rightarrow [0, \infty]$  is a measure. Let  $f$  be nonnegative and measurable on  $X$ . Then define  $\mu_f : \mathcal{A} \rightarrow [0, \infty]$  by

$$\mu_f(A) = \int_A f d\mu = \int_X f \cdot 1_A d\mu$$

Then  $\mu_f$  is a measure.

**Proposition 3.9.** Suppose that  $f$  is nonnegative and measurable on  $X$ , and suppose that  $\int_X f d\mu < \infty$ . Then the set  $\{x \in X \mid f(x) = \infty\}$  has measure 0.

**Proposition 3.10.** Suppose that  $X$  is a set,  $\mathcal{A}$  is a  $\sigma$ -algebra of subsets of  $X$  and  $\mu : \mathcal{A} \rightarrow [0, \infty]$  is a measure. Suppose that there is a set  $N \in \mathcal{A}$  with  $\mu(N) = 0$  and suppose that some property  $P$  holds for all  $x \in X$  outside  $N$ . Then we say that the property  $P$  holds almost everywhere or for almost all  $x \in X$ .

**Proposition 3.11.** *Suppose that  $f_k$  is a nonnegative measurable function on  $X$  for  $k = 0, 1, \dots$ . Suppose that*

$$\sum_{k=0}^{\infty} \left( \int_X f_k d\mu \right) < \infty$$

Then

$$\sum_{k=0}^{\infty} f_k < \infty \text{ for almost all } x \in X$$

**Proposition 3.12.** *Suppose that  $f$  is a nonnegative measurable function on  $X$ . Then*

$$\int_X f d\mu = 0 \iff f(x) = 0 \text{ almost everywhere.}$$

**Theorem 3.13** (Fatou's Lemma). *Suppose that  $f_k$  is a nonnegative measurable function on  $X$ , for  $k = 0, 1, \dots$ , and that  $\lim_{k \rightarrow \infty} f_k(x) = f(x)$  for each  $x \in X$ . Then*

$$\int_X f d\mu \leq \liminf_{k \rightarrow \infty} \left( \int_X f_k d\mu \right)$$

### 3.2. Integration of real and complex valued functions.

**Lemma 3.14.** *Let  $f : X \rightarrow \overline{\mathbb{R}}$  be a measurable function, and let  $f^+$  and  $f^-$  be the positive and negative parts of  $f$ . Then  $f = f^+ - f^-$ , and  $|f| = f^+ + f^-$ . Moreover,  $|f|$  is a measurable function, and*

$$f \text{ is integrable if and only if } \int_X |f| d\mu < \infty$$

**Definition 3.15** (Integral of a complex valued function). *Let  $f = u + iv$ , where  $u, v : X \rightarrow \overline{\mathbb{R}}$ . Then*

$$\int_X f d\mu = \int_X u d\mu + i \int_X v d\mu$$

**Lemma 3.16.** *Let  $f : X \rightarrow \overline{\mathbb{C}}$ . Then  $f$  is integrable if and only if  $\int_X |f| d\mu < \infty$ .*

The next theorem is probably the most important single theorem in these notes. It has many applications, both of a theoretical and practical nature.

**Theorem 3.17** (Dominated convergence theorem). *Let  $(f_k)$  be a sequence of real or complex valued measurable function on  $X$ . Assume that*

- $\lim_{k \rightarrow \infty} f_k(x) = f(x)$

and that there is a measurable function  $g : X \rightarrow [0, \infty]$  such that

- $|f_k(x)| \leq g(x)$  for each  $k$  and  $x$ , and
- $\int_X g d\mu < \infty$

Then

$$\int_X f d\mu = \lim_{k \rightarrow \infty} \int_X f_k d\mu$$



**Theorem 3.18** (Bounded convergence theorem). *Let  $(f_k)$  be a sequence of real or complex valued measurable function on  $X$ . Assume that*

- $\lim_{k \rightarrow \infty} f_k(x) = f(x)$  for each  $x \in X$ ,
- There exists a constant  $M < \infty$  such that  $|f_k(x)| \leq M$  for each  $k$  and  $x$ ,
- $\mu(X) < \infty$ .

Then

$$\int_X f \, d\mu = \lim_{k \rightarrow \infty} \int_X f_k \, d\mu$$

**Theorem 3.19.** *Suppose that  $f_k$  is a measurable real or complex valued function on  $X$  for  $k = 0, 1, \dots$ . Suppose that we have*

$$\sum_{k=0}^{\infty} \left( \int_X |f_k| \, d\mu \right) < \infty$$

or, equivalently,

$$\int_X \left( \sum_{k=0}^{\infty} |f_k| \right) \, d\mu < \infty$$

Then we have

$$\int_X \left( \sum_{k=0}^{\infty} f_k \right) \, d\mu = \sum_{k=0}^{\infty} \left( \int_X f_k \, d\mu \right)$$

**Definition 3.20** (Integrable function). We call  $f : X \rightarrow \mathbb{K}$   $\mu$ -integrable if  $f$  is  $\mu$  measurable and

$$\int_X |f| \, d\mu < \infty$$

We set

$$\mathcal{L}^1(X, \mathbb{K}) = \{f : X \rightarrow \mathbb{K} \mid f \mu\text{-integrable}\}$$

**Theorem 3.21.**  $\mathcal{L}^1(X, \mathbb{K})$  is a vector space over  $\mathbb{K}$

**Definition 3.22** (The Lebesgue-Stieltjes integral). Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be an increasing right continuous function, that is  $\lim_{s \rightarrow t^+} F(s) = F(t)$  for all  $t \in \mathbb{R}$ . Then for  $A \subseteq \mathbb{R}$  let

$$\mu_F^*(A) = \inf \left\{ \sum_{k=0}^{\infty} (F(b_k) - F(a_k)) \mid A \subseteq \bigcup_{k \in \mathbb{N}} (a_k, b_k) \right\}$$

The  $\mu_F^*$  is an outer measure inducing an inner measure on  $\mathbb{R}$ . Then we have

- $\mu_F$  is a Borel measure.
- $\mu_F((a, b]) = F(b) - F(a)$ .

We then define  $\int_A f \, dF = \int_A f \, d\mu_F$  as the Lebesgue-Stieltjes integral.

**Lemma 3.23.** *If  $\mu$  is a finite measure on  $\mathbb{R}$ , then we define  $F(t) = \mu((-\infty, t])$  as the **distribution function** of  $\mathbb{R}$ .*

**Theorem 3.24.** *There is a bijection from finite measures and some class of right-continuous increasing functions.*

**Definition 3.25** (Measures from other measures). Let  $g : X \rightarrow [0, \infty]$  be a  $\mu$ -measurable function. For  $A \in \mathcal{A}$  define

$$\nu(A) = \int_A g \, d\mu$$

Then using the monotone convergence theorem one can show that  $\nu$  is a measure defined on  $\mathcal{A}$ . Moreover, if  $f : X \rightarrow \mathbb{K}$  is  $\mu$ -measurable, then

$$\int_X f \, d\nu = \int_X fg \, d\mu$$

We call  $g$  the **density of  $\nu$  with respect to  $\mu$** .

**Proposition 3.26.** *Let  $f \in \mathcal{L}^1(X, \mathbb{K})$  with respect to the Lebesgue measure. Then*

$$\int_a^b f(x) \, dx = \lim_{t \rightarrow b^-} \int_a^t f(x) \, dx$$

### 3.3. Parameter integrals.

**Definition 3.27** (Parameter integral). Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $Y$  a metric space. Suppose that  $f : X \times Y \rightarrow \mathbb{K}$  is such that

- $x \mapsto f(x, y)$  is  $\mu$ -integrable for all  $y \in Y$ ,
- $y \mapsto f(x, y)$  is continuous at  $y_0$  for almost all  $x \in X$ ,
- there exists  $g \in \mathcal{L}^1(X, \mathbb{R})$  such that

$$|f(x, y)| \leq g(x)$$

for almost all  $x \in X$ .

Define  $F(y) = \int_X f(x, y) \, d\mu(x)$ . Then  $F$  is continuous at  $y_0 \in Y$ .

**Theorem 3.28** (Differentiation of parameter integrals.). *Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $L \subset \mathbb{R}$  an interval. Suppose that  $f : X \times L \rightarrow \mathbb{R}$  is such that*

- $x \mapsto f(x, y)$  is  $\mu$ -integrable for all  $y \in Y$ ,
- $\frac{\partial}{\partial t} f(x, t)$  exists for all  $t \in L$ , for almost all  $x \in X$ , and is continuous ,
- there exists  $g \in \mathcal{L}^1(X, \mathbb{R})$  with  $|\frac{\partial}{\partial t} f(x, t)| < g(x)$  for almost all  $x \in X$  and all  $t \in L$ .

Define  $F(t) = \int_X f(x, t) \, d\mu(x)$ . Then  $f : L \rightarrow \mathbb{K}$  is differentiable and

$$F'(t) = \int_X \frac{\partial}{\partial t} f(x, t) \, d\mu(x)$$

4. THE  $L^p$ -SPACES

**Definition 4.1** ( $L^p$ -spaces). Let  $1 \leq p < \infty$  and  $f : X \rightarrow \mathbb{K}$  measurable. We call

$$\|f\|_p = \left( \int_X |f|^p d\mu \right)^{\frac{1}{p}}$$

the  $L^p$ -norm of  $f$ . We set

$$\mathcal{L}^p(X) = \{f : X \rightarrow \mathbb{K} \mid f \text{ measurable, } \|f\|_p < \infty\}$$

**Theorem 4.2** (Hölder's inequality). Let  $p, q \in [1, \infty]$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ . If  $f \in \mathcal{L}^p(X)$  and  $g \in \mathcal{L}^q(X)$ , then

$$\left| \int_X fg d\mu \right| \leq \|f\|_p \|g\|_q$$

**Proposition 4.3** (Minkowski's inequality). If  $f, g \in \mathcal{L}^p$ ,  $1 \leq p \leq \infty$ , then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

**Definition 4.4** ( $L^p$ -spaces). Let  $f \sim g$  if  $f = g$  almost everywhere. Denote the equivalence class of  $f$  by  $[f]$ . Then

$$L^p(X) = \{[f] \mid f \in \mathcal{L}^p(X)\}$$

**Definition 4.5** (Cauchy sequence). A sequence  $(f_n) \in L^p(X)$  is called **Cauchy** if for every  $\epsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that

$$\|f_n - f_m\|_p < \epsilon$$

for all  $n, m > n_0$ .

**Theorem 4.6** (Completeness of  $L^p(X)$ ). Let  $(f_n)$  be a sequence in  $L^p(X)$ . Then  $(f_n)$  converges in  $L^p(X)$  if and only if  $(f_n)$  is a Cauchy sequence.

*Remark.* Introducing the metric  $d(f, g) = \|f - g\|_p$ , we have that  $L^p(X)$  is a **complete normed space** or a **Banach space**. If  $p = 2$ , then  $\|f\|_2$  is induced by an inner product - hence  $L^2(X)$  is a **complete inner product space**, or a **Hilbert space**.

**Proposition 4.7.** Suppose that  $f_n, f \in \mathcal{L}^p(X)$  with  $\|f_n - f\| \rightarrow 0$ . Then there exists a subsequence  $(f_{n_k})$  with  $f_{n_k}$  converging pointwise to  $f$  for almost every  $x \in X$ .

**Theorem 4.8.** The simple functions are dense in  $L^p(X)$  for  $1 \leq p < \infty$ .

In  $\mathbb{R}^N$  and the Lebesgue measure, we can modify the statement to the simple function with bounded support are dense in  $\mathbb{R}^N$ .

**Theorem 4.9.** For  $1 \leq p < \infty$

$$\text{span}\{1_U \mid U \subseteq \mathbb{R}^N \text{ open and bounded}\}$$

is dense in  $L^p(\mathbb{R}^N)$ .

We can also use bounded rectangles in the place of open bounded sets here.

**Definition 4.10** (Essential supremum). Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $f : X \rightarrow \mathbb{R}$   $\mu$ -measurable. We call  $\text{ess-sup} f(x) = \inf\{t \in \mathbb{R} \mid \mu(\{x \in X \mid f(x) > t\}) = 0\}$  the *essential supremum* of  $f$ . The essential supremum of  $|f|$  is denoted  $\|f\|_\infty$

**Theorem 4.11** (Completeness of  $L^\infty(X)$ ).  $L^\infty(X)$  is a complete normed space.

**Lemma 4.12.** Hölder's inequality holds for  $p = 1, q = \infty$ . That is,

$$\left| \int_X fg \, d\mu \right| \leq \|f\|_p \|g\|_q$$

**Lemma 4.13.** If  $\mu(X) < \infty$ , then  $\lim_{p \rightarrow \infty} \|u\|_p = \|u\|_\infty$

**4.1. Fubini's Theorem.**

**Theorem 4.14** (Tonelli). Suppose that  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow [0, \infty]$  is measurable. Then there exist sets  $N \subseteq \mathbb{R}^n$  and  $M \subseteq \mathbb{R}^m$  of measure zero such that

- (i)  $x \mapsto f(x, y)$  is measurable for all  $y \in \mathbb{R}^m - M$ ,
- (ii)  $y \mapsto \int_{\mathbb{R}^n} f(x, y) \, dx$  is measurable,
- (iii)  $y \mapsto f(x, y)$  is measurable for all  $x \in \mathbb{R}^n - N$ ,
- (iv)  $x \mapsto \int_{\mathbb{R}^m} f(x, y) \, dy$  is measurable,
- (v)

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^m} f(x, y) \, d(x, y) &= \\ \int_{\mathbb{R}^m - M} \left( \int_{\mathbb{R}^n} f(x, y) \, dx \right) \, dy &= \\ \int_{\mathbb{R}^n - N} \left( \int_{\mathbb{R}^m} f(x, y) \, dy \right) \, dx & \end{aligned}$$

**Theorem 4.15** (Fubini). Suppose that  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow [0, \infty]$  is measurable. Let  $N, M$  be the sets from Theorem 4.14 applied to the function  $|f|$  such that (v) holds with  $f$  replaced with  $|f|$ . Assume that one of these integrals is finite - and hence all of them. Then there exists sets  $N_1$  of  $\mathbb{R}^n$  and  $M_1$  of  $\mathbb{R}^m$  such that (i) - (v) of Theorem 4.14 hold with  $N, M$  replaced with  $N_1, M_1$ .

**Definition 4.16** (Complete measure space). Let  $(X, \mathcal{A}, \mu)$  be a measure space. We call the measure  $\mu$  *complete* if whenever  $A \in \mathcal{A}$  has measure 0, then any subset of  $A$  is in  $\mathcal{A}$ , (and has measure 0).

**Definition 4.17** ( $\sigma$ -finite measure space).

5. CONVOLUTION

**Definition 5.1** (Translation of a function). Let  $f : \mathbb{R}^N \rightarrow \mathbb{C}$  be a function and  $t \in \mathbb{R}^N$  a fixed vector. We define the translation operator  $\tau_t$  by

$$\tau_t f(x) = f(x - t)$$

**Theorem 5.2** (Continuity of translation). Let  $1 \leq p < \infty$  and  $f \in L^p(\mathbb{R}^N)$ . Then

$$\lim_{t \rightarrow 0} \|\tau_t f - f\|_p = 0$$

*Remark.* This does not hold if  $p = \infty$ .

**Lemma 5.3.** Let  $f : \mathbb{R}^N \rightarrow \mathbb{C}$  be measurable and set

$$\begin{aligned} F_1(x, y) &= f(x) \\ F_2(x, y) &= f(y - x) \end{aligned}$$

Then  $F_1, F_2 : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{C}$  are measurable.

**Definition 5.4** (Convolution). Let  $f, g : \mathbb{R}^N \rightarrow \mathbb{C}$  be measurable. We define the **convolution**  $f \star g : \mathbb{R}^N \rightarrow \mathbb{C}$  by

$$(f \star g)(x) = \int_{\mathbb{R}^N} f(x - y)g(y) dy$$

whenever the integral exists

**Definition 5.5** (Convex function). A function  $\varphi : (a, b) \rightarrow \mathbb{R}$  is called **convex** if

$$\varphi(\lambda s + (1 - \lambda)t) \leq \lambda\varphi(s) + (1 - \lambda)\varphi(t)$$

for all  $s, t \in (a, b)$  and all  $\lambda \in (0, 1)$

**Lemma 5.6.** This is equivalent to the condition

$$\frac{\varphi(t) - \varphi(s)}{t - s} \leq \frac{\varphi(u) - \varphi(t)}{u - t}$$

whenever  $a < s < t < u < b$ .

**Theorem 5.7** (Jensen's inequality in  $\mathcal{L}^p(X)$ -spaces). Let  $f \in \mathcal{L}^p(X)$ ,  $1 \leq p < \infty$ , and let  $g \in \mathcal{L}^1(X)$ . Then

$$\left( \int_X |fg| d\mu \right)^p \leq \|g\|_1^{p-1} \int_X |f|^p |g| d\mu$$

**Theorem 5.8** (Young's inequality). Let  $1 \leq p \leq \infty$ . If  $f \in L^p(\mathbb{R}^N, \mathbb{C})$  and  $g \in L^1(\mathbb{R}^N, \mathbb{C})$ , then  $f \star g$  exists almost everywhere and  $f \star g \in L^p(\mathbb{R}^N, \mathbb{C})$ . Moreover,

$$\|f \star g\|_p \leq \|f\|_p \|g\|_1$$

**Theorem 5.9.** Let  $1 \leq p, q \leq \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . If  $f \in L^p(\mathbb{R}^N)$  and  $g \in L^q(\mathbb{R}^N)$ , then

$$f \star g \in BC(\mathbb{R}^N)$$

where  $BC$  is the vector space of bounded continuous functions.

**5.1. Approximate identities.**

**Definition 5.10** (Approximate identity). Let  $\varphi : \mathbb{R}^N \rightarrow [0, \infty)$  be measurable with

$$\int_{\mathbb{R}^N} \varphi dx = 1$$

and set  $\varphi_n(x) = n^N \varphi(nx)$  for all  $x \in \mathbb{R}^N$  and  $n \in \mathbb{N}$ . Then  $(\varphi_n)$  is called an **approximate identity**

**Theorem 5.11.** Let  $(\varphi_n)$  be an approximate identity and  $f \in L^p(\mathbb{R}^N)$ ,  $1 \leq p < \infty$ . Then

$$f \star \varphi_n \rightarrow f$$

in  $L^p(\mathbb{R}^N)$  as  $n \rightarrow \infty$

**Theorem 5.12.** Let  $f \in L^\infty(\mathbb{R}^N)$  and  $(\varphi_n)$  an approximate identity. If  $f$  is continuous at  $x$ , then

$$f(x) = \lim_{n \rightarrow \infty} (f \star \varphi_n)(x)$$

**Definition 5.13** (Test function). Let  $U \subseteq \mathbb{R}^N$  be open. We let

$$C^\infty(U, \mathbb{K}) = \{f : U \rightarrow \mathbb{K} \mid f \text{ has partial derivatives of all orders}\}$$

and

$$C_c^\infty(U, \mathbb{K}) = \{f \in C^\infty(U, \mathbb{K}) \mid \text{supp}(f) \subseteq U, \text{supp}(f) \text{ compact}\}$$

The functions in  $C_c^\infty(U, \mathbb{K})$  are called **test functions** on  $U$ .

**Proposition 5.14.** Let  $f : \mathbb{R}^N \rightarrow \mathbb{K}$  be measurable such that  $f \in \mathcal{L}^1(B)$  for every bounded set  $B \subseteq \mathbb{R}^N$ . If  $\varphi \in C_c^\infty(\mathbb{R}^N)$ , then  $f \star \varphi \in C^\infty(\mathbb{R}^N)$  and

$$\frac{\partial}{\partial x_i} (f \star \varphi) = f \star \frac{\partial \varphi}{\partial x_i}$$

**Theorem 5.15.** Let  $U \subseteq \mathbb{R}^N$  open and  $1 \leq p < \infty$ . Then  $C_c^\infty(U)$  is dense in  $L^p(U)$ .

*Remark.* The above proposition does not hold for  $p = \infty$ .

**6. THE FOURIER TRANSFORM**

**Definition 6.1** (Fourier transform). Let  $f \in L^1(\mathbb{R}^N, \mathbb{C})$ . We call

$$\widehat{f}(t) = \int_{\mathbb{R}^N} f(x) e^{-2\pi i x \cdot t} dx$$

**Theorem 6.2.** We have

- $\widehat{f} : \mathbb{R}^N \rightarrow \mathbb{C}$  is continuous,
- $\|\widehat{f}\|_\infty \leq \|f\|_1$

**Proposition 6.3.** Let  $\varphi(x) = e^{-\pi|x|^2}$ . Then  $\|\varphi\|_1 = 1$  and  $\widehat{\varphi} = \varphi$ .

**Proposition 6.4.** Let  $f \in L^1(\mathbb{R}^N, \mathbb{C})$ ,  $x_0 \in \mathbb{R}^N$  and  $\alpha \in \mathbb{R}, \alpha > 0$ .

- (i) If  $g(x) = f(x - x_0)$ , then  $\widehat{g}(t) = e^{-2\pi i x_0 \cdot t} \widehat{f}(t)$ ,
- (ii) If  $g(x) = f(\alpha x)$ , then  $\widehat{g}(t) = \frac{1}{\alpha^N} \widehat{f}\left(\frac{t}{\alpha}\right)$ ,
- (iii) If  $g(x) = \overline{f(-x)}$ , then  $\widehat{g}(t) = \overline{\widehat{f}(t)}$

**Definition 6.5.** Let  $C_0(\mathbb{R}^N, \mathbb{K}) = \{f \in C(\mathbb{R}^N, \mathbb{K}) \mid \lim_{|x| \rightarrow \infty} f(x) = 0\}$ , the set of continuous functions vanishing at infinity

**Theorem 6.6** (Riemann-Lebesgue). If  $f \in L^1(\mathbb{R}^N, \mathbb{C})$ , then  $\widehat{f} \in C_0(\mathbb{R}^N, \mathbb{C})$  and  $\|\widehat{f}\|_\infty \leq \|f\|_1$ .

**Theorem 6.7.** If  $f, g \in L^1(\mathbb{R}^N, \mathbb{C})$  then  $f \star g \in L^1(\mathbb{R}^N, \mathbb{C})$  and

$$\widehat{f \star g} = \widehat{f} \widehat{g}$$

**Proposition 6.8.** Let  $f, g \in L^1(\mathbb{R}^N, \mathbb{C})$ . Then

$$\int_{\mathbb{R}^N} \widehat{f} \widehat{g} dx = \int_{\mathbb{R}^N} f g dx$$

**Lemma 6.9.** Let  $\varphi(x) = e^{-\pi|x|^2}$  and  $\varphi_n(x) = n^N \varphi(nx)$ . Then

$$\int_{\mathbb{R}^N} \widehat{f}(t) e^{2\pi i x \cdot t} \varphi\left(\frac{t}{n}\right) dt = (f \star \varphi_n)(x)$$

for all  $f \in L^1(\mathbb{R}^N, \mathbb{C})$ ,  $x \in \mathbb{R}^N$ , and  $n \in \mathbb{N}$ .

**Theorem 6.10** (Fourier inversion formula). Let  $f \in L^1(\mathbb{R}^N, \mathbb{C})$ . Then

(i)

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \widehat{f}(t) e^{2\pi i x \cdot t} e^{-\pi \frac{|t|^2}{n^2}} dt = f$$

in  $L^1(\mathbb{R}^N, \mathbb{C})$ .

(ii) If  $f$  is continuous at  $x$ , then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \widehat{f}(t) e^{2\pi i x \cdot t} e^{-\pi \frac{|t|^2}{n^2}} dt = f(x)$$

**Corollary.** Let  $f, g \in L^1(\mathbb{R}^N)$  with  $\widehat{f} = \widehat{g}$ . Then  $f = g$  almost everywhere.

**6.1. The Fourier transform on  $L^2(\mathbb{R}^N)$ .** We have defined the Fourier transform  $\widehat{f}$  with  $f \in L^1(\mathbb{R}^N)$ . We have that  $C_c^\infty(\mathbb{R}^N)$  is dense in  $L^2(\mathbb{R}^N)$  as well as in  $L^1(\mathbb{R}^N)$ , so in particular  $L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$  is dense in  $L^2(\mathbb{R}^N)$ . We can use this to extend the Fourier transform to  $L^2(\mathbb{R}^N)$ . The key for doing so is the following theorem.

**Theorem 6.11** (Plancherel). *Let  $f \in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ . Then  $\|\widehat{f}\|_2 = \|f\|_2$ .*

**Proposition 6.12.** *There is a unique continuous linear operator*

$$\mathcal{F} : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$$

*such that  $\mathcal{F}f = \widehat{f}$  for all  $f \in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ . Moreover,  $\|f\|_2 = \|\mathcal{F}f\|_2$  for all  $f \in L^2(\mathbb{R}^N)$ .*

*Remark.* We use the notation  $\widehat{f} = \mathcal{F}f$  for  $f \in L^2(\mathbb{R}^N)$ .

*Remark.* Let  $\varphi_n : \mathbb{R}^N \rightarrow [0, 1]$  such that  $\varphi_n \in L^2(\mathbb{R}^N)$  and  $\varphi_n(x) \rightarrow 1$  for all  $x \in \mathbb{R}^N$ . If  $f \in L^2(\mathbb{R}^N)$ , then

$$\widehat{f} = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(x)\varphi_n(x)e^{-2\pi i x \cdot t} dx$$

Common choices for  $\varphi_n$  are

- $\varphi_n(x) = 1_{B(0,n)}$ ,
- $\varphi_n(x) = e^{-\pi \frac{|x|^2}{n^2}}$

**Theorem 6.13.** *Let  $f \in L^2(\mathbb{R}^N, \mathbb{C})$ . Then*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \widehat{f}(t)e^{2\pi i x \cdot t} e^{-\pi \frac{|t|^2}{n^2}} dt = f$$

*in  $L^2(\mathbb{R}^N, \mathbb{C})$ .*

**Theorem 6.14.**  $\mathcal{F} : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$  *is bijective with  $(\mathcal{F}^{-1}f)(x) = (\mathcal{F}f)(-x)$  for all  $f \in L^2(\mathbb{R}^N)$ .*

*Remark.* Let

$$\langle f, g \rangle = \int_{\mathbb{R}^N} f(x)\overline{g(x)} dx$$

denote the inner product on  $L^2(\mathbb{R}^N)$ . Then the above theorem implies

$$\langle \widehat{f}, \widehat{g} \rangle = \langle f, g \rangle.$$

Moverover, by approximating  $f, g$  by functions in  $L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$  we also have

$$\langle \widehat{f}, g \rangle = \langle f, \widetilde{g} \rangle$$

## 7. THE RADON-NIKODYM THEOREM

**7.1. The Riesz representation theorem.** Let  $H$  be an inner product space with inner product  $(x|v)$ . Then  $H$  is a normed space with norm

$$\|u\| = \sqrt{(u|u)}$$

We call  $H$  a **Hilbert space** if  $H$  is complete with respect to  $\|\cdot\|$ , that is, every Cauchy sequence in  $H$  converges.



**Theorem 7.1** (Projections). *Let  $H$  be a Hilbert space and  $M \subseteq H$  a closed subspace of  $H$ . Let  $u \in H$ . Then there exists  $m_0 \in M$  such that*

$$\|u - m_0\| = \min_{m \in M} \|u - m\|$$

Moreover,

$$(u - m_0 | m) = 0$$

for all  $m \in M$ .

*Remark.* Fix  $g \in H$  and consider the function  $\varphi_g : H \rightarrow \mathbb{K}$  given by

$$\varphi_g(f) = (f | g)$$

Then  $\varphi_g$  is linear, and by the Cauchy-Swartz inequality,

$$|\varphi_g(f)| = |(f | g)| \leq \|f\| \|g\|$$

for all  $f \in H$ . We say  $\varphi_g$  is a bounded linear functional on  $H$ .

**Definition 7.2.** Let  $H$  be a Hilbert space. We call a linear operator  $\varphi : H \rightarrow \mathbb{K}$  a **bounded linear functional** on  $H$  if there exists  $M > 0$  such that

$$|\varphi(f)| \leq M \|f\|$$

for all  $f \in H$ .

**Theorem 7.3** (Riesz representation theorem). *Let  $H$  be a Hilbert space over  $\mathbb{K}$  and  $\varphi : H \rightarrow \mathbb{K}$  a bounded linear function. Then there exists  $g \in H$  such that*

$$\varphi(f) = (f | g)$$

for all  $f \in H$ .

**7.2. The Radon-Nikodym Theorem.** Suppose that  $\mu$  is a measure defined on the  $\sigma$ -algebra  $\mathcal{A}$  of subsets of  $X$ . Given a measurable function  $g : X \rightarrow [0, \infty]$  we define

$$\nu(A) = \int_A g d\mu$$

Then  $\nu$  is a measure defined on the  $\sigma$ -algebra  $\mathcal{A}$ .

The converse does not necessarily hold - that is, given two measures  $\mu$  and  $\nu$  on a  $\sigma$ -algebra  $\mathcal{A}$ , there is not always a measurable function  $g : X \rightarrow [0, \infty]$  such that the above equation holds.

**Definition 7.4** (Absolute continuity). Let  $\nu, \mu$  be the measures defined on a  $\sigma$ -algebra  $\mathcal{A}$ . We call  $\nu$  **absolutely continuous with respect to  $\mu$**  if  $\nu(A) = 0$  whenever  $\mu(A) = 0$ . In that case, we write  $\nu \ll \mu$ .

**Proposition 7.5.** Let  $\mu, \nu$  be measures defined on a  $\sigma$ -algebra. Suppose that  $\mu(X), \nu(X) < \infty$ . Set  $\lambda = \mu + \nu$ . Then there exists a measurable function  $h : X \rightarrow [0, \infty]$  such that

$$\int_X f d\nu = \int_X fh d\lambda$$

for all  $f \in L^2(X, \lambda)$

**Theorem 7.6** (Radon-Nikodym). Let  $\mu, \nu$  be measures defined on a  $\sigma$ -algebra. Suppose that  $\nu$  and  $\mu$  are  $\sigma$ -finite and that  $\nu \ll \mu$ . Then there exists a measurable function  $g : X \rightarrow [0, \infty)$  such that  $\nu(A) = \int_A g d\mu$  for all  $A \in \mathcal{A}$ .

Formally we can write

$$\int_X f d\nu = \int_X f \frac{d\nu}{d\mu} d\mu$$

if we define  $g = \frac{d\nu}{d\mu}$ , where  $g$  is the density function from the Radon-Nikodym theorem.

*Remark.* If  $g$  is the function in the Radon-Nikodym theorem, it is not hard to show that

$$\int_X f d\nu = \int_X fg d\mu$$

for all  $f \in L^1(X, \nu)$ .

## 8. PROBABILITY THEORY

**Definition 8.1** (Random variable). Let  $(\Omega, \mathcal{A}, P)$  be a probability space. A  $\mathcal{A}$ -measurable function

$$X : \Omega \rightarrow \mathbb{R}$$

is called a **random variable**.

**Definition 8.2.** Let  $X : \Omega \rightarrow \mathbb{R}$  a random variable. We say that  $X$  has *finite expectation* if  $X \in L^1(\Omega)$  and call

$$E[X] = \int_{\Omega} X dP$$

the **expectation** of  $X$ .

**Definition 8.3** (Distribution). For every Borel set  $A \subseteq \mathbb{R}$  we define

$$P_X[A] = P[\{\omega \in \Omega | X(\omega) \in A\}] = P[X \in A]$$

Since  $X$  is measurable,  $X^{-1}[A]$  is measurable for all Borel sets  $A \subseteq \mathbb{R}$ .

**Definition 8.4** (Distribution). Let  $X$  be a random variable. The Borel measure defined above is called the **distribution** of  $X$ . The function

$$F(t) = P_X [(-\infty, t]] = P[X \leq t]$$

is called the **distribution function** of  $X$ .

**Lemma 8.5.** *Let  $X$  be a random variable and let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be Borel measurable. Then*

$$\int_{\Sigma} f \circ X dP = \int_{\mathbb{R}} f dP_X$$

### 8.1. Conditional expectation.

**Definition 8.6** (Conditional expectation). Let  $X : \Omega \rightarrow \mathbb{R}$  a random variable with finite expectation. Let  $\mathcal{A}_0$  be a  $\sigma$ -algebra with  $\mathcal{A}_0 \subseteq \mathcal{A}$ . We call

$$X_0 : \Sigma \rightarrow \mathbb{R}$$

a **conditional expectation** given  $\mathcal{A}_0$  if

- $X_0$  is  $\mathcal{A}_0$ -measurable
- $\int_A X_0 dP = \int_A X dP$  for all  $A \subseteq \mathcal{A}_0$ .

We write  $X_0 = E[X|\mathcal{A}_0]$

**Theorem 8.7.** *Let  $X$  be a random variable with finite expectation. If  $\mathcal{A}_0$  is a  $\sigma$ -algebra with  $\mathcal{A}_0 \subseteq \mathcal{A}$ , then the conditional expectation  $X_0 = E[X|\mathcal{A}_0]$  exists and is essentially unique.*

*Remark.* • If  $X$  is  $\mathcal{A}_0$ -measurable, then  $X = E[X|\mathcal{A}_0]$  almost everywhere.

- If we set  $\mathcal{A}_0 = \{\varphi, \Omega\}$ , then

$$E[X|\mathcal{A}_0] = E[X]$$