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RAMSAY THEORY

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1

Monochromatic Systems

Theorem 1.1 (Ramsay's theorem)**.** *Whenever $\mathbb{N}^{(2)}$ is two-coloured, there exists an infinite monochromatic set.*

(i) Called a "two-pass" proof.

(ii) Same proof that whenever $N^{(2)}$ is k-coloured. Alternatively, view color as 1 and "2 or 3 or ... or k". and by theorem one we get an infinite set of colour 1 - then just induct on k.

(iii) Having an infinite monochromatic set is stronger than asking for an arbitrarily large finite monochromatic set.

Example 1.2. *Any sequence $x_1, x_2, \ldots$ in $\mathbb{R}$ (or any totally ordered set) has a monotone subsequence.*

Proof. Color **up** if $x_i < x_j$, **down** if $x_i \geq x_j$, and apply Theorem 1.1. □

What about $\mathbb{N}^{(r)}, r = 3, 4, \ldots$. If we two-color $\mathbb{N}^{(r)}$, can we get an infinite monochromatic set?

For example, consider $n = 3$. Color $N^{(3)}$ by colouring (i, j, k) **red** if i divides $j + k$, **blue** if not.

Theorem 1.3 (Ramsey's theorem for r-sets)**.** *Whenever $\mathbb{N}^{(r)}$ is two-coloured, there exists an infinite monochromatic set.*

Proof. Induction on r. $r = 1$ is trivial by the pigeonhole principle. $r = 2$ is shown by Theorem 1.1.

Now, given a two-colouring of $N^{(r)}$. Choose $a_1 \in \mathbb{N}$. We induce a two-colouring c' of $(\mathbb{N} - \{a_1\})^{(r-1)}$ by $c'(F) = c(F \cup \{a_1\})$ for all $F \in$

$(\mathbb{N} - \{a_1\})^{(r-1)}$. By induction, there exists an infinite monochromatic set $B_1 \subseteq N - \{a_1\}$ for c'.

So all r-sets $F \cup \{a_1\}$, $F \subset B_1$ have the same color (c_1, say). Choose $a_2 \in B_1$. By the same argument, there exists an infinite set $B_2 \subset B_1 - \{a_2\}$ such that all r-sets $F \cup \{a_2\}$, $F \subset B_2$ have the same colour. Continue inductively. We obtain a sequence of points $a_1, a_2, \dots$ and colors $c_1, c_2, \dots$ such that each r-set $a_{i_1}, \dots, a_{i_r}$ with $i_1 < \dots < i_2$ has color c_{i_1}. But we must have $c_{i_1} = c_{i_2} = c_{i_3} = \dots$ for some infinite subsequence. Then $\{a_{i_1}, a_{i_2}, \dots\}$ is an infinite monochromatic sequence. $\square$

Example 1.4. *We can show that given any* $(1, x_1), (2, x_2), \dots$ *we can find a subsequence inducing a monotone function. Consider the three-colouring of* $(1, x_1), (2, x_2), (3, x_3), \dots$ *by colouring triples of points* ***convex*** *or* ***convex*** *depending on the colouring of the set.*

Theorem 1.5. *Infinite Ramsey (Theorem 1.3) implies the finite version. That is, for all* $m, r \in \mathbb{N}$*, whenever* $[m]^{(r)}$ *is two-coloured there exists a monochromatic* m*-set.*

Proof. Suppose not, so for all $n \geq r$ there exists a two-colouring c_n of $[m]^{(r)}$ without a monochromatic m-set. We'll construct a 2-colouring of $\mathbb{N}^{(r)}$ without a monochromatic m-set, contradicting Theorem 1.3.[1]

There are only finitely many ways to two-color $[r]^{(r)}$ (two, in fact). So infinitely many of the c_n agree on $[r+1]^{(r)}$. Say, $c_i | [r+1]^{(r)} = d_{r+1}$. Now,

(i) the d_i are nested, and

(ii) no d_n has a monochromatic m-set (as there is some k such that $d_n = c_k | [n]^{(r)}$.

Define a colouring $c : \mathbb{N}^{(r)} \to [2]$ by $c(F) = d_n(F)$ for any $n \geq \max F$. We obtain our contradiction. $\square$

[1] If the c_n nested - that is, if $c_n|_{[n-1]^{(r)}} = c_{n-1}$, can take union, but they may **not** be nested

Remark 1.6. *(i) Proof gives no bound on what* $n = n(m, r)$ *we could take. There* ***are*** *direct proofs that do give upper bounds.*

(ii) Called a "compactness argument". Essentially, we are proving that the space $[0, 1]^{\mathbb{N}}$ *(all infinite 0-1 sequences) with the product topology (e.g. the metric* $d(f, g) = \frac{1}{\min(n : f_n \neq g_n)}$ *is (sequentially) compact.*

What if we coloured $\mathbb{N}^{(2)}$ with ∞ many colours (i.e. w have $c : \mathbb{N}^{(2)} \to X$ for some set X). Obviously, we cannot find an infinite M on which c is constant - for example, let c be injective.

Can we always find an infinite M such that c is either constant on $M^{(2)}$ or injective on $M^{(2)}$? No - for example, $1 \mapsto \{2, 3, 4, \dots\}, 2 \mapsto \{3, 4, 5, \dots\}, \dots$ as different colours as a counterexample.

Theorem 1.7 (Canonical Ramsey Theorem)**.** *Let $c : \mathbb{N}^{(2)} \to X$ for some set X. Then there exists an infinite $M \in \mathbb{N}$ such that one of the following holds:*

(i) *c is constant on $M^{(2)}$,*

(ii) *c is injective on $M^{(2)}$,*

(iii) *$c(i, j) = c(k, l) \iff i = k$ with $(i, j, k, l \in M, i < j, k < l)$*

(iv) *$c(i, j) = c(k, l) \iff j = l)$ with $(i, j, k, l \in M, i < j, k < l)$*

Remark 1.8. *This generalizes enormously Theorem 1.1 - if X is finite then (i), (iii), (iv) cannot arise.*

Proof. We'll apply this for Ramsey's theorem on 4-sets. Two-colour $\mathbb{N}^{(4)}$ by giving (i, j, k, l) colour **same** if $c(i, j) = c(k, l)$, **different** otherwise.

By Ramsey's theorem for 4-sets (Theorem 1.3), there exists an infinite set M_1 that is monochromatic for this colouring.

If M_1 is coloured **same**, for any i, j and k, l in M_1^2, choose $m, n \in M_1^{(2)}$ within $m > j, l$. Then $c(i, j) = c(m, n)$ and $c(k, l) = c(m, n)$. So $c(i, j) = c(k, l)$ so c is constant on $M_1^{(2)}$.

So now, we may assume M_1 is coloured differently. Now two-colour $M_1^{(4)}$ by giving (i, j, k, l) **same** if $c(i, l) = c(j, k)$, **different** otherwise. By Theorem 1.3, there exists an infinite set $M_2 \subset M_1$ that is monochromatic for this colouring.

If M_2 are coloured the same, choose $i < j < k < l < m < n$ in M_2. Then $c(j, k) = c(i, n)$ and $c(l, m) = c(i, n)$, whence $c(j, k) = c(l, m)$, which is a contradiction, as $M_2 \subset M_1$. Thus, M_2 is coloured **different**.

Two-colour $M_2^{(4)}$ by giving (i, j, k, l) colour **same** if $c(i, k) = c(j, l)$, **different** otherwise. We have an infinite monochromatic colouring

$M_3 \subset M_2$ for this colouring. If M_3 coloured **same**, choose $i < j < k < l < m < n$ in M_3. Then $c(i,l) = c(j,m)$ and $c(i,l) = c(k,m)$, so $c(j,n) = c(k,m)$, a contradiction. So M_3 is coloured **different**.

Two-colour $M_3^{(3)}$ by giving (i,j,k) colour **same** if $c(i,j) = c(j,k)$, **different** otherwise. We have an infinite monochromatic sequence $M_4 \subset M_3$ for this colouring. If M_4 is coloured same, choose $i < j < k < l$ in M_4. Then $c(i,j) = c(j,k) = c(k,l)$, a contradiction. So, M_4 is coloured **differently**.

Two-colour $M_4^{(3)}$, by giving (i,j,k) colour **left-same** if $c(i,j) = c(i,k)$, **left-different** otherwise. We have an infinite monochromatic set $M_5 \subset M_4$ for this. Then two-colour $M_5^{(3)}$ by giving (i,j,k) colour **right-same** if $c(j,k) = c(i,k)$, **right-different** if not. We get an infinite monochromatic sequence M_6 for this colouring.

If M_6 is **left-different, right-different**, we have case (iii). If M_6 is **left-same, right-different**, we have case (ii). If M_6 is **left-different, right-same**, we have case (iv). If M_6 is **left-same, right-same**, choosing $i < j < k$ in M_6, then $c(i,j) = c(i,k) = c(j,k)$, which is a contradiction. $\square$

Remark 1.9. *(i) Could use just one colouring, according to the pattern of colours on the 2-sets inside a given 4-set.*

*(ii) For any r, one can show similarly. For **any** colouring c of $\mathbb{N}^{(r)}$, there exists an infinite $M \subset \mathbb{N}$ and $I \subset [r]$ such that for all $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_r$ in M, $c(i_1, \ldots, i_r) = c(j_1, \ldots, j_r) \iff i_n = j_n$ for all $n \in I$. These 2^r colourings are the canonical colourings of $\mathbb{N}^{(r)}$.*

For example, let $r = 2$. $I = \{1\}$ is case (iii). $I = \{2\}$ is case

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Van Der Waerden's Theorem

Theorem 2.1. *Whenever $\mathbb{N}$ is two-coloured, there exists a monochromatic arithmetic progression of length m, for any $m \in \mathbb{N}$.*

Theorem 2.2. *Let $k \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that whenever $[n]$ is 2-coloured, there exists a monochromatic arithmetic progression of length m.*

One idea in the proof is - we show that $\forall m, k \in \mathbb{N}$, whenever $[n]$ is k-coloured, there exists a monochromatic arithmetic progression of length m.[1]

Write $W(m, k)$ for the least such n (if it exists) - a "Wan Der Waerden's number". Let $A_1, \ldots, A_r$ be arithmetic progressions of length $m - 1$, say

$$A_i = \{a_i, a_i + d_i, \ldots, a_i + (m-2) * d_i\} \tag{2.1}$$

We say $A_1, \ldots, A_r$ are **focused** at f if $a_i + (m-1)d_i = f$ for all i - for example, $\{1, 4\}$ and $\{5, 6\}$ are focused at 7. If each A_i are monochromatic (for a given colouring), with no two A_i the same colour, say that $A_i, \ldots, A_r$ are colour-focused at f.[2]

Proposition 2.3. *Let $k \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that whenever $[n]$ is k-coloured, there exists a monochromatic arithmetic progression of length 3.*[3]

Lemma 2.4. *We claim the following result - for all $r \leq k$, there exists n such that whenever $[n]$ is k-coloured, there exists a monochromatic arithmetic progression of length 3 or there exist r colour-focused arithmetic progressions of length 2.*

[1] Harder results **could** be easier to prove - if the proof is by induction!

[2] So if we have a r-colouring, and $A_1, \ldots, A_r$ are colour-focused. Then, we get a monochromatic arithmetic progression of length m - by asking, what colour is the focus f.

[3] This will be subsumed by Van Der Waerden's theorem

Proof. Proceed by induction on r. This is true for $r = 1$ (setting $n = k + 1$.) We'll show that if n is suitable for $r - 1$ then

$$(k^{2n} + 1) \cdot 2n \tag{2.2}$$

is suitable for r. Indeed, given a k-colouring of $[(k^{2n} + 1)2n]$ with no monochromatic arithmetic progression of length 3.

Break up $[(k^{2n} + 1)2n]$ into intervals $B_1, \dots, B^{k^{2n}+1}$ of length $2m$ - so $B_i = [2m(i-1) + 1, 2ni]$ for $i = 1, 2, \dots, k^{2n} + 1$.Now, there are k^{2m} ways to k-colour a block. Thus, there exist two blocks coloured identically - say B_s and B_{s+t}.

... □

Complete this proof

Missed lecture...

Theorem 2.5 (Strengthened Van Der Warden)**.** *Let $m \in \mathbb{N}$. Then whenever $\mathbb{N}$ is finitely coloured there exists an arithmetic progression that (together with it's common difference) is monochromatic.*

Proof. Induction on k, the number of colours. Given n suitable for $k - 1$ (whenever $[n]$ is $k - 1$ coloured there exists a monochromatic arithmetic progression with common difference of length n), then $W(n(m-1) + 1, k)$ is suitable for k.

Given k-colouring of $[W(n(m-1) + 1, k)]$, there exists a monochromatic arithmetic progression of length $n(m-1) + 1$ - say $a, a + d, a + 2d, \dots, a + n(m-1)d$. If d or $2d$ or $\dots$ is the same color as the arithmetic progression, we are done. Otherwise, $\{d, 2d, \dots, nd\}$ is $k - 1$coloured, so we are done by induction. □

Remark 2.6. *(i) Henceforth, we do not care about bounds.*

(ii) The case $k = 2$ is Shur's theorem - whenever $\mathbb{N}$ is finitely coloured, there exist x, y, z monochromatic with $x + y = z$.

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The Hales-Jewett Theorem

Let X be a finite set. Subset of X^n is a **line** or **combinatorial line** if there exists $I \subset [n]$, $I \neq \emptyset$, and $a_i \in X$ for each $i \in [n] - I$, such that

$$L = \{x \in X^n | x_i = a_i \forall i \notin I, x_j = x_k forall j, k \in I\} \tag{3.1}$$

Theorem 3.1. *Let $m, k \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that whenever $[m]^n$ is k-coloured there exists a monochromatic line.*

Remark 3.2. *(i) The smallest such n is denoted $HJ(m, k)$.*

(ii) So m-in-a-row naughts and crosses played in enough dimensions cannot end in a draw.[1]

(iii) Hales-Jewett implies Van Der Waerden's theorem. Indeed, given a k-colouring on $\mathbb{N}$, induce a k-colouring of $[m]^n$ by $c'((x_1, x_2, \ldots, x_n) = c(x_1 + x_2 + \cdots + x_n)$. By Hales-Jewett, there exists a monochromatic line for c'.

[1] Exercise: show that first-player winers.

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Bibliography