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# PERCOLATION

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# Contents

# 1

# Random Walks on Graphs

Our basic setting is the (hyper-)cubic lattice on $\mathbb{R}^d, d \geq 1$. This is the graph with vertex set $\mathbb{Z}^d$, edges $\langle x, y \rangle \iff \|x - y\|_1 = 1$, and edge set denoted $E^d$. A lattice is $L^d = (\mathbb{Z}^d, E^d)$.

## 1.1 Percolation

Let $0 < p < 1$. Let $e \in E^d$, and with probability $p$ independently for each edge, declare $e$ to be **open** else **closed**. Consider $x \leftrightarrow y$ if there exists an open path from $x$ to $y$. The **open cluster** at $x$ is $C_x = \{y : x \leftrightarrow y\}$.

**Theorem 1.1**

*For a given $p$, what can be said about the $C_x$?*

For $p = 1$, $C_x = Z_d$. For $p = 0$, $C_x = \{x\}$.

**Definition 1.2** (Percolation probability)**.** Let $\theta(p) = \mathbb{P}(|C_\theta|) = \mathbb{P}_p$. Note that $\theta$ is non-decreasing.

Let $p_c = \sup\{p : \theta(p) = 0\}$.

It is known that $\theta$ is $C^\infty$ on $(p_c, 1]$, and that $\theta$ is right-continuous on $[0, 1]$.

It is believed that $\theta$ is concave on $(p_c, 1]$, and that $\theta$ is real-analytic on $(p_c, 1]$, and that $\theta(p_c) = 0$ (known for $d = 2$, and $d \geq 16$).

**Definition 1.3.** Probability theorey. Let $\Omega = \{0, 1\}^{E^d}$, $\mathcal{F}$ be the $\sigma$-filed generated by the finite-dimensional cylinder ... of form . $\{\omega \in \Omega : \omega = \xi \text{ on } \mathcal{F}\} = E_F(\xi)$. ...

Fill in from lecture notes

**Theorem 1.4.** *For $d \geq 2$, $0 < p_c < 1$.*

Fill in lecture notes from Chapter 3 of Probability on Graphs

Consider $\mathbb{Z}^d$, with $\kappa_n = \mu^{n(1+o(1))}$ as $n \to \infty$, $\mu = \mu(\mathbb{Z}^d)$

We have $\kappa_n \sim An^c\mu^n$ for some $A = A(d), c = c(d)$ where $a_n \sim b_n$ means $\frac{a_n}{b_n} \to 1$.

$c_n$ is called the **critical exponent**. People are hoping to show that for $d = 2$, $c = \frac{11}{32}$. $c$ is expected to be **universal** in that it depends on $d$ but not each $d$-dimensional graph.

## 1.2 Coupling

Let $L^d = (\mathbb{Z}^d, E^d)$ consider $P_p$ on $\Omega = \{0,1\}^{E^d}$.

Let $(U_e, e \in E)$ be independent uniform random variables $U(0,1)$.

Let $p \in (0,1)$. Then

$$\mu_p(e) = \begin{cases} 0 & U_e \geq p \\ 1 & U_e < p \end{cases} \tag{1.1}$$

if $p_1 \leq p_2$ then $\mu_{p_1}(e) \leq \mu_{p_2}(e)$.

$\mu_p : 0 < p < 1$ is a coupling of percolations, containing all interesting, "universal" in $p$.

**Theorem 1.5.** *For any increasing function $f : \Omega \to \mathbb{R}$,*

$$\mathbb{E}_{p_1}(f) \leq \mathbb{E}_{p_2}(f) \tag{1.2}$$

*for $p_1 \leq p_2$.*

**Example 1.6.** *For example, $u, v \in \mathbb{Z}^d$, $f(u) = \mathbb{I}(u \leftrightarrow v)$. Then $\mathbb{P}_{p_1}(u \leftrightarrow v) \leq \mathbb{P}_{p_2}(u \leftrightarrow v)$.*

## 1.3 Oriented/Directed Percolations

Consider the standard percolation, and define

$$\overrightarrow{\theta}(p) = \mathbb{P}(\text{there exists an infinite directed path through the origin to } p). \tag{1.3}$$

Then $\overrightarrow{p_c} = \sup\{p : \overrightarrow{\theta}(p) = 0\}$. As $\overrightarrow{\theta}(p) \leq \theta(p)$, we have $\overrightarrow{p_c} \geq p_c$.

## 1.4 Correlation Inequalities

Consider a set $E$ be nonempty and finite, and $\Omega = \{0,1\}^E$. The sample space $\Omega$ is partially ordered by $\omega_1 \leq \omega_2$ if $\omega_1(e) \leq \omega_2(e)$ for all $e \in E$.

Event $A \subseteq \Omega$ is called **increasing** if $w \in A, w \leq w' \Rightarrow w' \in A$ and decreasing if $\overline{A} = \Omega \backslash A$ is increasing.

**Definition 1.7.** With two probability measures $\mu_1, \mu_2$, we write $\mu_1 \leq_{st} \mu_2$ if $\mu_1(A) \leq \mu_2(A)$ for all increasing events $A$.

Equivalently, $\mu_1 \leq_{st} \mu_2$ if and only if $\mu_1(f) = \sum_\Omega f(\omega)\mu_1(\omega) \leq \mu_2(f)$ for all increasing functions $f : \Omega \to \mathbb{R}$.

Let $S \subseteq \Omega^2$ given by $S = \{(\pi, \omega) \in \Omega^2 : \pi \leq \omega\}$.

**Theorem 1.8** (Strassen)**.** *The following are equivalent:*

*(i)* $\mu_1 \leq \mu_2$

*(ii) There exists a probability measure $\kappa$ on $\Omega^2$ such that*

  *(i)* $\kappa(S) = 1$

  *(ii) Marginals of $\kappa$ are $\mu_1$ and $\mu_2$.*

*Proof.* From reference in the back. □

**Theorem 1.9** (Holley's inequality)**.** *Let $\mu_1, \mu_2$ be probability measures which are positive (in that $\mu_i(\omega) > 0$ for all $i$ and $\omega \in \Omega$). If*

$$\mu_2(\omega_1 \vee \omega_2)\mu_1(\omega_1 \wedge \omega_2) \geq \mu_1(\omega_1)\mu_2(\omega_2) \tag{1.4}$$

*for all $\omega_1, \omega_2 \in \Omega$, then $\mu_1 \leq \mu_2$.*

*The notation is*

$$(\omega_1 \vee \omega_2)(e) = \max \omega_1(e), \omega_2(e) \tag{1.5}$$

$$(\omega_1 \wedge \omega_2)(e) = \min \omega_1(e), \omega_2(e) \tag{1.6}$$

*Proof.* See Probability and Random Processes (Stirzaker), the section on Markov chains in continuous time for the necessary background.

*Definition* 1.10 (Markov Chains). $(X_t, t \geq 0)$ taking values in a state space $S$, which is finite satisfying the Markov property

*Definition* 1.11 (Markov Property). For all $x, y \in S, x \neq y$,

$$\mathbb{P}(X_{t+h} = y | X_t = x) = hG(x, y) + o(h) \tag{1.7}$$

as $h \downarrow 0$

The matrix $G = (G(x,y))_{x,y \in S}$ is the **generator** of the Markov chain. The diagonal elements $G(x,x)$ are chosen such that the row sums are all zero,

$$\sum_{y \in S} G(x, y) = 0 \tag{1.8}$$

for all $x \in S$.

*Definition* 1.12 (Invariant distribution). $\pi$ on $S$ is an invariant distribution if it satisfies if $X_0$ has distribution $\pi$, then $X_t$ has distribution $\pi$ for all $t \geq 0$.

*Lemma* 1.13. *$\pi$ is invariant if and only if $\pi G = 0$.*

*Definition* 1.14. $X$ is **time reversible** if $\pi(x)G(x,y) = \pi(y)G(y,x)$ for all $x, y \in S$ where $\pi$ is (say) invariant.

If detailed balance holds for some $\pi$ then $\pi$ is invariant.

Let $\mu$ be a positive probability measure on $\Omega$.

For $\omega \in \Omega$ and $e \in E$, define the configurations $\omega^e, \omega_e$ by

$$\omega^e(f) = \begin{cases} w(f) & f \neq e \\ 1 & f = e \end{cases} \tag{1.9}$$

$$\omega^e(f) = \begin{cases} w(f) & f \neq e \\ 0 & f = e \end{cases} \tag{1.10}$$

Let $G : \Omega^2 \to \mathbb{R}$ be given by $G(\omega_e, \omega^e) = 1$, $G(\omega^e, \omega_e) = \frac{\mu(\omega_e)}{\mu(\omega^e)}$, for all $\omega \in \Omega$ and $e \in E$. Set $G(w, w') = 0$ for all other elements (coordinate distance greater than 2), and $G(w, w) = -\sum_{\omega' \neq \omega} G(\omega, \omega')$.

$G$ is the generator for a Markov chain $X$ on $\Omega$. We then have

$$\mu(\omega)G(\omega, \omega') = \mu(\omega')G(\omega', w) \tag{1.11}$$

(trivial from the construction of $G$).

Thus, $\mu$ is invariant for $X$.

Now, construct a Markov chain $((X_t, Y_t)t \geq 0)$ taking values in $S = \Omega^2$. Let $\mu_1, \mu_2$ be positive probability measures on $\Omega$, assumed positive.

Let $G$ be given by

$$G((\pi_e, \omega), (\pi^e, \omega^e)) = 1 \tag{1.12}$$

$$G((\pi, \omega^e), (\pi_e, \omega_e)) = \frac{\mu_2(\omega_e)}{\mu_2(\omega^e)} \tag{1.13}$$

$$G((\pi^e, \omega^e), (\pi_e, \omega^e)) = \frac{\mu_1(\pi_e)}{\mu_1(\pi_e)} - \frac{\mu_2(\omega_e)}{\mu_2(\omega^e)} \geq 0 \tag{1.14}$$

from the conditions of the theorem.

Defining $G(x, y) = 0$ otherwise and $G(x, x)$ to satisfy the zero row-sum condition, we have that $G$ is a Markov chain. Thus it has an invariant measure $\mu$. Then $X$ is a Markov chain, having measure $\mu_1$. $Y$ is a Markov chain, having invariant measure $\mu_2$.

Then by Strassen's theorem ( 1.8), $\mu_1 \leq \mu_2$.

Choose $\mu_1, \mu_2$ satisfying the condition. Let $Z = (X, Y)$ a Markov chain on $\Omega^2$, in fact on $S = \{(\pi, \omega) : \pi \leq \omega\}$.

$X$ is a Markov chain with invarnaitn measure $\mu_1$. $Y$ is a Markov chain with invariant distirbuiton $\mu_2$.

Then $Z$ has an invariant measure $\kappa$ on $S$. Let $f : \Omega \to \mathbb{R}$ be increasing. Then $\mu_1(f) = \kappa(f(\pi)) \leq \kappa(f(\omega)) = \mu_2(f)$.

This completes the proof. □

**Theorem 1.15** (FKG inequality)**.** *Let $\mu$ be a probability measure on $\Omega = \{0,1\}^E$ with $|E| < \infty$ such that $\mu$ is positive and*

$$\mu(\omega_1 \vee \omega_2)\mu(\omega_1 \wedge \omega_2) \geq \mu(\omega_1)\mu(\omega_2), \tag{1.15}$$

*known as the **FKG lattice condition**.*

*Then $\mu$ is **positively associated** in that*

$$\mu(fg) \geq \mu(f)\mu(g) \tag{1.16}$$

*for all increasing random variables $f, g : \Omega \to \mathbb{R}$ or equivalently,*

$$\mu(A \cap B) \geq \mu(A)\mu(B) \tag{1.17}$$

*for all increasing events $A, B$.*

**Example 1.16.** *Consider a percolation, with $A = \{x \leftrightarrow\}$, $B = \{u \leftrightarrow v\}$. Then we have*

$$\mathbb{P}_p(x \leftrightarrow | u \leftrightarrow v) \geq \mathbb{P}_p(x \leftrightarrow y). \tag{1.18}$$

**History 1.17.** *When $\mu$ is a product measure, this was first proven by Harris (1961) by induction on $|E|$.*

*Proof.* Let $\mu_1 = \mu$. Note that (1.16) is invariant under $g \mapsto g + c$, for $c \in \mathbb{R}$. Thus we may assume that $g$ is strictly positive. Then

$$\mu_2(\omega) = \frac{\mu(\omega) g(\omega)}{\sum_{w'} g(\omega') \mu(\omega')} \tag{1.19}$$

Since $g$ is increasing, $\mu_1 \leq \mu_2$ follows by the FKG lattice condition. By the Holley inequality, $\mu_1(f) \leq \mu_2(f)$ for $f$ increasing. Therefore, $\mu(f) \leq \frac{\mu(fg)}{\mu(g)}$ as required. □

### 1.4.1 *The BK Inequality*

Consider $\Omega = \{0,1\}^E, |E| < \infty$. Let $\omega \in \Omega, F \subseteq E$. The consider

$$C(\omega, F) = \{w' \in \Omega : \omega'(e) = \omega(e) \forall e \in F\} = (w(e) : e \in F) \times \{0,1\}^{E \setminus F} \tag{1.20}$$

Let $A, B \subseteq \Omega$. Then define

$$A \square B = \{\omega \in \Omega : \exists F \subseteq E, C(\omega, F) \subseteq A, C(\omega, \overline{F}) \subseteq B\} \subseteq A \cap B. \tag{1.21}$$

If $A, B$ are increasing, then $C(\omega, F) \subseteq A$ if and only if $\omega_F \in A$, where

$$\omega_F(e) = \begin{cases} w(e) & e \in F \\ 0 & e \notin F \end{cases} \tag{1.22}$$

In this case $A \square B = \{\omega : \exists F \subseteq E s.t. \omega_F \in A, \omega_{E \setminus F} \in B\}$.

**Theorem 1.18** (BK inequality)**.** *For increasing subsets For product measure $\mathbb{P}$ (say $\mathbb{P}_{p_e}(w(e) = 1)$ for some given $(p_e, e \in E)$),*

$$\mathbb{P}(A \square B) \leq \mathbb{P}(A)\,\mathbb{P}(B) \tag{1.23}$$

*for all increasing events $A, B$.*

**Theorem 1.19** (Reimer's inequality)**.**

$$\mathbb{P}(A \square B) \leq \mathbb{P}(A)\,\mathbb{P}(B) \tag{1.24}$$

*for all $A, B \subseteq \Omega$ and product measures $\mathbb{P}$.*

## 1.5 Influence

**Question 1.20.** *What is the influence of an individual in an election?*

**Question 1.21.** *An increasing event $A$, a sequence of measures $\mathbb{P}_p$, and consider $g(p) = \mathbb{P}_p(A)$.*

For example, consider a problem from reliability theory - an electrical network has every link cut with probability $1 - p$, and what is the probability that the network is still connected? This class of theorems are called "$S$-shaped theorems".

$\Omega = \{0, 1\}^E$, $|E| < \infty$, $|E| = N$, $A \subseteq \Omega$.

Let $e \in E$.

**Definition 1.22.** The influence of $e$ on $A$ is

$$I_A(e) = \mathbb{P}_p(\mathbb{I}(A)\,(\omega^e) \neq \mathbb{I}_A(\omega_e)). \tag{1.25}$$

If $A$ is increasing, then

$$I_A(e) = \mathbb{P}_p(A^e) - \mathbb{P}_p(A_e). \tag{1.26}$$

where

$$A^e = \{\omega : \omega^e \in A\} \tag{1.27}$$

$$A_e = \{\omega : \omega_e \in A\} \tag{1.28}$$

$$\tag{1.29}$$

**Theorem 1.23** (Kahn-Kalani-Limial, Talagrand). *There exists $c > 0$ such that for all $\epsilon$, $A$ and $0 < p < 1$. Then*

$$\sum_{e \in E} I_A(e) \geq c[\mathbb{P}_p(A)\mathbb{P}_p(\overline{A})] \log \frac{1}{\max_{e \in E} I_A(e)}. \tag{1.30}$$

*Proof.* One uses discrete Fourier analysis (but non-examinable). □

**Theorem 1.24.** *It is interesting if we have uniform upper bound $M_p$ for the $I_A(e)$.*

*Let $m = \max_{e \in E} I_A(e)$. Then we can write*

$$mN \geq [\cdots] \log \frac{1}{m} \tag{1.31}$$

$$m \geq \frac{[\cdots]}{N} \log \frac{1}{m} \geq [\cdots]' \frac{\log N}{N}. \tag{1.32}$$

**Theorem 1.25** (Restatement of KKL). *The maximum influence $M$ satisfies*

$$m \geq c' \mathbb{P}_p(A)\mathbb{P}_p(\overline{A}) \frac{\log N}{N} \tag{1.33}$$

*for some universal $c' > 0$.*

*The $\frac{\log N}{N}$ is optimal.*

**Example 1.26** (Tribes). *Consider $N$ people partitioned into $t$ tribes, each of size $s = \log N - \log \log N + \alpha$, and let $p = \frac{1}{2}$.*

*Then let*

$$A = \{\text{There exists a tribe all of whose elements are 1}\} \tag{1.34}$$

*Then*

$$I_A(e) \sim c\mathbb{P}(A)\,\mathbb{P}(\overline{A})\,\frac{\log N}{N} \tag{1.35}$$

*for all $e$.*

**Theorem 1.27** (Symmetric Case). *If $I_A(e)$ is a constant for $e \in E$,*

$$\sum_{e \in E} I_A(e) \geq c[\mathbb{P}_p(A)\mathbb{P}_p(\overline{A})] \log N. \tag{1.36}$$

## 1.6 Sharp Threshold

Let $\Omega$ as before, $A \subseteq \Omega$. Then

**Theorem 1.28** (Rousseau, Margoulis).

$$\frac{d}{dp}\mathbb{P}_p(A) = \sum_{e \in E} \mathbb{P}_p(A^e) - \mathbb{P}_p(A^e). \tag{1.37}$$

*Note this is equal to $\sum_{e\in E} I_A(e)$ if $A$ is increasing.*

*Proof.* Need to only consider

$$\mathbb{P}_p(A) = \sum_{\omega} \mathbb{I}(A)\,(\omega) p^{|\eta|}(1-p)^{N-|\eta|} \tag{1.38}$$

where $N = |E|$, $\eta = \{e : \omega(e) = 1\}$.

Then

$$\frac{d}{dp}\mathbb{P}_p(A) = \sum_{\omega} \mathbb{I}(A)\,(\omega)(\frac{|\eta|}{p} - \frac{N-|\eta|}{1-p})p^{|\eta|}(1-p)^{|N-|\eta|} \tag{1.39}$$

and so

$$\begin{aligned} p(1-p)\frac{d}{dp}\mathbb{P}_p(A) &= \sum_{\omega} \mathbb{I}(A)\,(\omega)(|\eta| - Np)p^{|\eta|}(1-p)^{N-|\eta|} && (1.40)\\ &= \mathbb{P}_p(\mathbb{I}(A)\,(|\eta| - Np)) && (1.41)\\ &= \sum_e \mathbb{P}_p(\mathbb{I}(A)\,(\mathbb{I}(e) - p)) && (1.42)\\ &= \sum_e \mathbb{P}_p(\mathbb{I}(A)\,\mathbb{I}(e)) - p\mathbb{P}_p(A) && (1.43)\\ &= \sum_e p\mathbb{P}_p(A^e) - p(p\mathbb{P}_p(A^e) + (1-p)\mathbb{P}_p(A_e)) && (1.44)\end{aligned}$$

where $\mathbb{I}(e) = \mathbb{I}(e \text{ open}) = \omega(e)$, so $|\eta| = \sum_e \mathbb{I}(e)$.

This completes the proof. □

## 1.7 Back to Percolation

Let $L^d = (\mathbb{Z}^d, \mathbb{E}^d), 0 < p < 1$, and measure $\mathbb{P}_p$. Let $N$ be the number of open clusters. Then

$$\mathbb{P}_p(N \geq 1) = \begin{cases} 0 & p < p_c \\ 1 & p > p_c \end{cases} \tag{1.45}$$

Then $\theta(p) = \mathbb{P}_p(0 \in \text{infinite open cluster})$. So

$$\theta = \begin{cases} 0 & p < p_c \\ > 0 & p > p_c \end{cases} \tag{1.46}$$

To show (**??**) implies (**??**), we have $\mathbb{P}_p(N \geq 1) \leq \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(x \in \text{infinite open cluster}) = \sum_x 0 = 0$.

To show (**??**) implies (**??**), by Kolmogrov's zero-one law, we have $\mathbb{P}_p(N \geq 1) \in \{0, 1\}$, but $\mathbb{P}_p(N \geq 1) \geq \theta(p) > 0$ for $p > p_c$.

**Theorem 1.29** (Uniqueness of infinite cluster). *For all* $0 < p < 1$, *either*

$$\mathbb{P}_p(N = 0) = 1 \tag{1.47}$$

*or*

$$\mathbb{P}_p(N = 1) = 1 \tag{1.48}$$

*Proof.* Fix $p \in (0, 1)$.

*Lemma* 1.30 (Part A). *There exists* $k = k_p \in \{0, 1, , 2, \dots, \} \cup \{\infty\}$ *with* $\mathbb{P}_p(N = k) = 1$.

*Proof.* Ł$^d$ comes equipped with a shift translation, and the measure is invariant under this shift. Thus $N = N(\omega)$ is invariant under the shift. □

This proof requires this lemma.

*Lemma* 1.31. *Any shift-invariant random variable on* $(\Omega, \mathcal{F}, \mathbb{P}_p)$ *is almost surely constant.*

*Proof.* Elementary application of measure theory. □

*Lemma* 1.32 (Part B). $k_p \in \{0, 1, \infty\}$ - *the "finite-energy property".*

*Proof.* Suppose $2 \leq k_p < \infty$.

Find $n$ such that $\mathbb{P}_p(\Lambda_n \text{intersections} \geq 2 \text{ infinite open clusters}) > \frac{1}{2}$. □

Follow this argument? p93 in Probability on Graphs

*Lemma* 1.33 (Part C). $k_p \neq \infty$.

*Proof.* Say $x$ is a trifurcation if

(i) $|C_x| = \infty$.

(ii) The removal of $x$ breaks $C_x$ into three disjoint infinite clusters.

Then $\tau = \mathbb{P}_p(x \text{ is a trifurcation})$ is independent of $x$.

We claim $\tau > 0$. To show this, take a large diamond box that intersects with at least three open clusters. Then there exists $n$ such that $\mathbb{P}_p(S_n \text{ intersects} \geq 3 \text{ infinite open clusters}) > \frac{1}{2}$.

Thus $\tau > 0$.

The argument is then that we use the ration beween boundary and volume to bound the number of trifuricatinos in $A_n$, and show that this leads to a contradiction for large $n$. □

Fill in rest of proof. Required to understand high-level ideas and key steps around the graphs

N.B. - consider the corresponding proof for site percolation. For $x, y, z \in \partial S$, does there exist open paths to zero? □

## 1.8 *Percolation in Two Dimensions*

There are two models, bond percolation on $L^2$, and site percolation on $\Pi$, the triangular lattice.

The triangular lattice is "self-matching", in that the dual construction is on the same lattice as the primal (c.f. the dual of the square lattice).

### 1.8.1 *Bond percolation on $\mathbb{Z}^2$, Site Percolation on $\mathbb{T}$*

**Theorem 1.34.** *For bond percolation on $\mathbb{Z}^2$, $\Theta(\frac{1}{2}) = 0$.*

*Proof* (Proof of Zhang). Let $p = \frac{1}{2}$ and suppose $\Theta(\frac{1}{2}) > 0$. Since $\Theta(\frac{1}{2}) > 0$, then the probability there exists an infinite open cluster is one.

Let $T_n = [0, n]^2$. As $n$ goes to infinity, then the probability that $T_n$ intersects with the infinite open cluster tends to one. Thus, find $N$ such that for all $n > N$, $\mathbb{P}(T_n \text{ intersects the infinite open cluster})$ is greater than $1 - \frac{1}{8}^4$.

Consider $A^t$ be the event that the **top** of $T_n$ is joined to the infinite open cluster. Define $A^b, A^l, A^r$ to be the **bottom**, **left**, and

**right** analogues. Then $\mathbb{P}(T_n$ does not intersects the infinite cluster) is $\mathbb{P}\left(\overline{A^t} \cap \overline{A^b} \cap \overline{A^l} \cap \overline{A^r}\right) \geq \mathbb{P}\left(\overline{A^u}\right)^4$ for $u = t, b, l, r$.

Then we have $\mathbb{P}(A^u) \geq \frac{7}{8}$ by the given result.

Let $n = N + 1$. Pass to the dual percolation, ...

$\square$

Fill this in from the Probability on Graphs book. Doesn't look too difficult.

### 1.8.2 *Site percolation on* $\Pi$

$\Pi$ has the vertex set $\{m\tilde{i} + n\tilde{j} : m, n \in \mathbb{Z}\}$, $\tilde{i} = (1, 0)$, $\tilde{j} = \frac{1}{2}(1, \sqrt{3})$ when embedded into $\mathbb{R}^2$.

Now, consider a box in $\mathbb{R}^2$, with vertices $(0, 0)$ and $(a, b)$ with $a \in \mathbb{N}, b \in \sqrt{3}2\mathbb{N}$.

Each site is black with probability $\frac{1}{2}$, and white otherwise. Let $H_{a,b} = \{L \leftrightarrow^{black} R \in R_{a,b}$ where $L$ is the left edge and $R$ is the right edge. That is, $H_{a,b}$ is the event that there exitss a black path that traverses $R_{a,b}$ from $L(R_{a,b})$ to $R(R_{a,b})$.

Then we have the lemma as follows:

**Lemma 1.35** (RSW Lemma)**.**

$$\mathbb{P}(H_{2a,b}) \geq \frac{1}{4}\mathbb{P}(H_{a,b})^2. \tag{1.49}$$

Proof on p100-105 in book

**Theorem 1.36.** $p_c(bond, \mathbb{T}) = \frac{1}{2}$

**Theorem 1.37.** $p_c \geq \frac{1}{2}$, *and in fact* $\Theta(\frac{1}{2}) = 0$ *for the bond model on* $\mathbb{Z}^2$.

*Proof.* Following p 122 of the book.

We need to prove that $p_c \leq \frac{1}{2}$ - that is $\Theta(p) > 0$ for $p > \frac{1}{2}$.

Let $H_n = H_{16n, n\sqrt{3}}$ be the event that a black crossing of $R_{16n, n\sqrt{3}}$ exists. By the previous lemma,s there exists $\tau > 0$ such that $\mathbb{P}_{\frac{1}{2}}(H_n) \geq \tau$ for some $\tau > 0$. Let $\frac{1}{2} \leq p \leq \frac{3}{4}$.

Then

$$(1-p)I_{n,p}(x) \leq \mathbb{P}_{1-p}(\text{Rad}(C_x) \geq n) \leq \mathbb{P}_{\frac{1}{2}}(\text{Rad}(C_0) \geq n) = \nu_n \to 0 \tag{1.50}$$

where $\text{Rad}(C_x) = \max\{|y - x| : x \leftrightarrow y\}$.

So we have

$$\frac{d}{dp}\mathbb{P}_p(H_n) \geq c\tau(1-\mathbb{P}_p(H_n))\log\frac{1}{8\nu_n} \tag{1.51}$$

and integrating gives

$$\int_{\frac{1}{2}}^{p}\frac{g'(p)}{1-g(p)}dp \geq c\tau\log\frac{1}{8\nu_n}(p-\frac{1}{2}) \tag{1.52}$$

and so $\mathbb{P}_p(H_n) \geq 1-(1-\tau)(8\nu_n)^{c\tau(p-\frac{1}{2})} \to 1$ as $n \to \infty$ if $p > \frac{1}{2}$.

... □

Fill in rest of proof (block argument)

## 1.9 *Cardy's Formula*

Given a Jordan curve on $R^2$, there exists a conformal map from $D$ to the interior of the equilateral triangle $T$ of $C$ with vertices $A = 0, B = 1, C = e^{\frac{\pi i}{3}}$ and such that $\phi$ can be extended to the boundary $\partial D$ in such a way that it becomes a homeomorphism from $D \cup \partial D$ to the closed triangle $T$.

**Theorem 1.38** (Cardy's Formula)**.**

$$\mathbb{P}_\delta(ac \leftrightarrow bx \text{ in } D) \to |BX| \tag{1.53}$$

*as* $\delta \to 0$.

Fill in the rest of the proof of Cardy's formula.

# 2

# Self Avoiding Walks

Consider $G$ a graph, with $\gamma$ is a self avoiding walk which visits each vertex of $G$ at most once.

$G_n(v)$ is the number of self avoiding walks with length $n$. We assume $G$ is transitive. Then $G_n$ is submultiplicative, and we defined

$$K(G) = \lim_{n\to\infty} \sqrt{G_n}n \tag{2.1}$$

For examples $K(d - ary - tree) = d - 1$, and $K(G) \leq \Delta - 1$ (exercise).

**Theorem 2.1.** $K(H1) = \sqrt{2+\sqrt{2}}$.

## 2.1 Generating Functions

$$Z(z) = \sum_{\gamma SAW}^{z^{|\gamma|}} \sum_{n=1}^{\infty} G_n \cdot z^n \tag{2.2}$$

Cauchy-Hadarmad gives the radius of convergence is $\frac{1}{\lim_{n\to\infty} \sqrt{G_n}n} = \frac{1}{K(G)}$

Fill in proof of $\sqrt{2+\sqrt{2}}$

## 2.2 Random Clusters Model/FK (Fortun-Kostelyn) Percolation

**Definition 2.2.** $G = (V, E)$, $\Omega = \{0,1\}^E$, for $\omega \in \Omega$, $k(\omega)$ is the number of open clusters.

The RC measure $p \in [0,1]$, $q \in (0,\infty)$,

$$\phi_{p,q}(w) = \frac{1}{Z_{p,q}} \prod_{e \in E} p(\omega(e))(1-p)^{1-\omega(e)})q^{k(w)} \tag{2.3}$$

(i) $q = 1$ is standard percolation,

(ii) $p, q \to 0$ with $\frac{q}{p} \to 0$ is electrical networks.

(iii) For $q = 2$, we have the FK Isiing model, where for $\omega \in \{0,1\}^E$, for each open cluster of $\omega$, we set the spins/states of the vertices of it to $\pm 1$ iwht equal probability, so $G \in \{\pm 1\}^V$, with

$$\mu_\beta(G) = \frac{1}{Z_\beta} \exp(\beta \sum_{x \sim y} G_x G_y) \tag{2.4}$$

and $p = 1 - e^{-\beta}$.

Our aim is to define the random cluser measure on $(Z^d, E^d)$. Let $\Lambda$ be a a finite box in $\mathbb{Z}^d$. Let $E_n = \{(u,v) \in E^{d|u,v \in \Lambda}$. Let $b = 0, 1$, and $\Omega_\Lambda^b = \{\omega \in \Omega = \{0,1\}^{E^d} | \omega(e) = b \forall e \notin \Lambda\}$.

Let $\phi^b_{\Lambda,p,q}(\omega) = \frac{1}{Z_{\lambda,p,q}} (\prod_{e \in E_n} p^{\omega(e)}(1-p)^{1-\omega(e)})q^{k(\omega,\Lambda)}$.

*3*

# *Bibliography*