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1

Random Walks on Graphs

Our basic setting is the (hyper-)cubic lattice on $\mathbb{R}^d, d \geq 1$. This is the graph with vertex set \mathbb{Z}^d , edges $\langle x, y \rangle \iff \|x - y\|_1 = 1$, and edge set denoted E^d . A lattice is $L^d = (\mathbb{Z}^d, E^d)$.

1.1 Percolation

Let $0 < p < 1$. Let $e \in E^d$, and with probability p independently for each edge, declare e to be **open** else **closed**. Consider $x \leftrightarrow y$ if there exists an open path from x to y . The **open cluster** at x is $C_x = \{y : x \leftrightarrow y\}$.

Theorem 1.1

For a given p , what can be said about the C_x ?

For $p = 1$, $C_x = \mathbb{Z}^d$. For $p = 0$, $C_x = \{x\}$.

Definition 1.2 (Percolation probability). Let $\theta(p) = \mathbb{P}(|C_\theta|) = \mathbb{P}_p$.

Note that θ is non-decreasing.

Let $p_c = \sup\{p : \theta(p) = 0\}$.

It is known that θ is C^∞ on $(p_c, 1]$, and that θ is right-continuous on $[0, 1]$.

It is believed that θ is concave on $(p_c, 1]$, and that θ is real-analytic on $(p_c, 1]$, and that $\theta(p_c) = 0$ (known for $d = 2$, and $d \geq 16$).

Definition 1.3. Probability theory. Let $\Omega = \{0, 1\}^{E^d}$, \mathcal{F} be the σ -field generated by the finite-dimensional cylinder ... of form $\{\omega \in \Omega : \omega = \zeta \text{ on } \mathcal{F}\} = E_{\mathcal{F}}(\zeta)$

Fill in from lecture notes

Theorem 1.4. For $d \geq 2$, $0 < p_c < 1$.

Consider \mathbb{Z}^d , with $\kappa_n = \mu^{n(1+o(1))}$ as $n \rightarrow \infty$, $\mu = \mu(\mathbb{Z}^d)$

We have $\kappa_n \sim An^c \mu^n$ for some $A = A(d), c = c(d)$ where $a_n \sim b_n$ means $\frac{a_n}{b_n} \rightarrow 1$.

c_n is called the **critical exponent**. People are hoping to show that for $d = 2$, $c = \frac{11}{32}$. c is expected to be **universal** in that it depends on d but not each d -dimensional graph.

Fill in lecture notes from
Chapter 3 of Probability on
Graphs

1.2 Coupling

Let $L^d = (\mathbb{Z}^d, E^d)$ consider P_p on $\Omega = \{0, 1\}^{E^d}$.

Let $(U_e, e \in E)$ be independent uniform random variables $U(0, 1)$.

Let $p \in (0, 1)$. Then

$$\mu_p(e) = \begin{cases} 0 & U_e \geq p \\ 1 & U_e < p \end{cases} \quad (1.1)$$

if $p_1 \leq p_2$ then $\mu_{p_1}(e) \leq \mu_{p_2}(e)$.

$\mu_p : 0 < p < 1$ is a coupling of percolations, containing all interesting, “universal” in p .

Theorem 1.5. For any increasing function $f : \Omega \rightarrow \mathbb{R}$,

$$\mathbb{E}_{p_1}(f) \leq \mathbb{E}_{p_2}(f) \quad (1.2)$$

for $p_1 \leq p_2$.

Example 1.6. For example, $u, v \in \mathbb{Z}^d$, $f(u) = \mathbb{I}(u \leftrightarrow v)$. Then

$$\mathbb{P}_{p_1}(u \leftrightarrow v) \leq \mathbb{P}_{p_2}(u \leftrightarrow v).$$

1.3 Oriented/Directed Percolations

Consider the standard percolation, and define

$$\vec{\theta}(p) = \mathbb{P}(\text{there exists an infinite directed path through the origin to } p). \quad (1.3)$$

Then $\vec{p}_c = \sup\{p : \vec{\theta}(p) = 0\}$. As $\vec{\theta}(p) \leq \theta(p)$, we have $\vec{p}_c \geq p_c$.

1.4 Correlation Inequalities

Consider a set E be nonempty and finite, and $\Omega = \{0,1\}^E$. The sample space Ω is partially ordered by $\omega_1 \leq \omega_2$ if $\omega_1(e) \leq \omega_2(e)$ for all $e \in E$.

Event $A \subseteq \Omega$ is called **increasing** if $w \in A, w \leq w' \Rightarrow w' \in A$ and decreasing if $\bar{A} = \Omega \setminus A$ is increasing.

Definition 1.7. With two probability measures μ_1, μ_2 , we write $\mu_1 \leq_{st} \mu_2$ if $\mu_1(A) \leq \mu_2(A)$ for all increasing events A .

Equivalently, $\mu_1 \leq_{st} \mu_2$ if and only if $\mu_1(f) = \sum_{\Omega} f(\omega)\mu_1(\omega) \leq \mu_2(f)$ for all increasing functions $f : \Omega \rightarrow \mathbb{R}$.

Let $S \subseteq \Omega^2$ given by $S = \{(\pi, \omega) \in \Omega^2 : \pi \leq \omega\}$.

Theorem 1.8 (Strassen). *The following are equivalent:*

- (i) $\mu_1 \leq \mu_2$
- (ii) *There exists a probability measure κ on Ω^2 such that*

- (i) $\kappa(S) = 1$

- (ii) *Marginals of κ are μ_1 and μ_2 .*

Proof. From reference in the back. □

Theorem 1.9 (Holley's inequality). *Let μ_1, μ_2 be probability measures which are positive (in that $\mu_i(\omega) > 0$ for all i and $\omega \in \Omega$). If*

$$\mu_2(\omega_1 \vee \omega_2)\mu_1(\omega_1 \wedge \omega_2) \geq \mu_1(\omega_1)\mu_2(\omega_2) \quad (1.4)$$

for all $\omega_1, \omega_2 \in \Omega$, then $\mu_1 \leq \mu_2$.

The notation is

$$(\omega_1 \vee \omega_2)(e) = \max \omega_1(e), \omega_2(e) \quad (1.5)$$

$$(\omega_1 \wedge \omega_2)(e) = \min \omega_1(e), \omega_2(e) \quad (1.6)$$

Proof. See Probability and Random Processes (Stirzaker), the section on Markov chains in continuous time for the necessary background.

Definition 1.10 (Markov Chains). $(X_t, t \geq 0)$ taking values in a state space S , which is finite satisfying the Markov property

Definition 1.11 (Markov Property). For all $x, y \in S, x \neq y$,

$$\mathbb{P}(X_{t+h} = y | X_t = x) = hG(x, y) + o(h) \quad (1.7)$$

as $h \downarrow 0$

The matrix $G = (G(x, y))_{x, y \in S}$ is the **generator** of the Markov chain. The diagonal elements $G(x, x)$ are chosen such that the row sums are all zero,

$$\sum_{y \in S} G(x, y) = 0 \quad (1.8)$$

for all $x \in S$.

Definition 1.12 (Invariant distribution). π on S is an invariant distribution if it satisfies if X_0 has distribution π , then X_t has distribution π for all $t \geq 0$.

Lemma 1.13. π is invariant if and only if $\pi G = 0$.

Definition 1.14. X is **time reversible** if $\pi(x)G(x, y) = \pi(y)G(y, x)$ for all $x, y \in S$ where π is (say) invariant.

If detailed balance holds for some π then π is invariant.

Let μ be a positive probability measure on Ω .

For $\omega \in \Omega$ and $e \in E$, define the configurations ω^e, ω_e by

$$\omega^e(f) = \begin{cases} w(f) & f \neq e \\ 1 & f = e \end{cases} \quad (1.9)$$

$$\omega_e(f) = \begin{cases} w(f) & f \neq e \\ 0 & f = e \end{cases} \quad (1.10)$$

Let $G : \Omega^2 \rightarrow \mathbb{R}$ be given by $G(\omega_e, \omega^e) = 1$, $G(\omega^e, \omega_e) = \frac{\mu(\omega_e)}{\mu(\omega^e)}$, for all $\omega \in \Omega$ and $e \in E$. Set $G(w, w') = 0$ for all other elements (coordinate distance greater than 2), and $G(w, w) = -\sum_{\omega' \neq w} G(w, \omega')$.

G is the generator for a Markov chain X on Ω . We then have

$$\mu(\omega)G(\omega, \omega') = \mu(\omega')G(\omega', \omega) \quad (1.11)$$

(trivial from the construction of G).

Thus, μ is invariant for X .

Now, construct a Markov chain $((X_t, Y_t)_{t \geq 0})$ taking values in $S = \Omega^2$. Let μ_1, μ_2 be positive probability measures on Ω , assumed positive.

Let G be given by

$$G((\pi_e, \omega), (\pi^e, \omega^e)) = 1 \quad (1.12)$$

$$G((\pi, \omega^e), (\pi_e, \omega_e)) = \frac{\mu_2(\omega_e)}{\mu_2(\omega^e)} \quad (1.13)$$

$$G((\pi^e, \omega^e), (\pi_e, \omega_e)) = \frac{\mu_1(\pi_e)}{\mu_1(\pi^e)} - \frac{\mu_2(\omega_e)}{\mu_2(\omega^e)} \geq 0 \quad (1.14)$$

from the conditions of the theorem.

Defining $G(x, y) = 0$ otherwise and $G(x, x)$ to satisfy the zero row-sum condition, we have that G is a Markov chain. Thus it has an invariant measure μ . Then X is a Markov chain, having measure μ_1 . Y is a Markov chain, having invariant measure μ_2 .

Then by Strassen's theorem (1.8), $\mu_1 \leq \mu_2$.

Choose μ_1, μ_2 satisfying the condition. Let $Z = (X, Y)$ a Markov chain on Ω^2 , in fact on $S = \{(\pi, \omega) : \pi \leq \omega\}$.

X is a Markov chain with invariant measure μ_1 . Y is a Markov chain with invariant distribution μ_2 .

Then Z has an invariant measure κ on S . Let $f : \Omega \rightarrow \mathbb{R}$ be increasing. Then $\mu_1(f) = \kappa(f(\pi)) \leq \kappa(f(\omega)) = \mu_2(f)$.

This completes the proof. \square

Theorem 1.15 (FKG inequality). *Let μ be a probability measure on $\Omega = \{0, 1\}^E$ with $|E| < \infty$ such that μ is positive and*

$$\mu(\omega_1 \vee \omega_2)\mu(\omega_1 \wedge \omega_2) \geq \mu(\omega_1)\mu(\omega_2), \quad (1.15)$$

known as the FKG lattice condition.

Then μ is positively associated in that

$$\mu(fg) \geq \mu(f)\mu(g) \quad (1.16)$$

for all increasing random variables $f, g : \Omega \rightarrow \mathbb{R}$ or equivalently,

$$\mu(A \cap B) \geq \mu(A)\mu(B) \quad (1.17)$$

for all increasing events A, B .

Example 1.16. Consider a percolation, with $A = \{x \leftrightarrow\}$, $B = \{u \leftrightarrow v\}$.

Then we have

$$\mathbb{P}_p(x \leftrightarrow | u \leftrightarrow v) \geq \mathbb{P}_p(x \leftrightarrow y). \quad (1.18)$$

History 1.17. When μ is a product measure, this was first proven by Harris (1961) by induction on $|E|$.

Proof. Let $\mu_1 = \mu$. Note that (1.16) is invariant under $g \mapsto g + c$, for $c \in \mathbb{R}$. Thus we may assume that g is strictly positive. Then

$$\mu_2(\omega) = \frac{\mu(\omega)g(\omega)}{\sum_{\omega'} g(\omega')\mu(\omega')} \quad (1.19)$$

Since g is increasing, $\mu_1 \leq \mu_2$ follows by the FKG lattice condition. By the Holley inequality, $\mu_1(f) \leq \mu_2(f)$ for f increasing. Therefore, $\mu(f) \leq \frac{\mu(fg)}{\mu(g)}$ as required. \square

1.4.1 The BK Inequality

Consider $\Omega = \{0, 1\}^E$, $|E| < \infty$. Let $\omega \in \Omega$, $F \subseteq E$. Then consider

$$C(\omega, F) = \{\omega' \in \Omega : \omega'(e) = \omega(e) \forall e \in F\} = (w(e) : e \in F) \times \{0, 1\}^{E \setminus F} \quad (1.20)$$

Let $A, B \subseteq \Omega$. Then define

$$A \square B = \{\omega \in \Omega : \exists F \subseteq E, C(\omega, F) \subseteq A, C(\omega, \bar{F}) \subseteq B\} \subseteq A \cap B. \quad (1.21)$$

If A, B are increasing, then $C(\omega, F) \subseteq A$ if and only if $\omega_F \in A$, where

$$\omega_F(e) = \begin{cases} w(e) & e \in F \\ 0 & e \notin F \end{cases} \quad (1.22)$$

In this case $A \square B = \{\omega : \exists F \subseteq E \text{ s.t. } \omega_F \in A, \omega_{E \setminus F} \in B\}$.

Theorem 1.18 (BK inequality). For increasing subsets For product measure \mathbb{P} (say $\mathbb{P}_{p_e}(w(e) = 1)$ for some given $(p_e, e \in E)$),

$$\mathbb{P}(A \square B) \leq \mathbb{P}(A) \mathbb{P}(B) \quad (1.23)$$

for all increasing events A, B .

Theorem 1.19 (Reimer's inequality).

$$\mathbb{P}(A \square B) \leq \mathbb{P}(A) \mathbb{P}(B) \quad (1.24)$$

for all $A, B \subseteq \Omega$ and product measures \mathbb{P} .

1.5 Influence

Question 1.20. What is the influence of an individual in an election?

Question 1.21. An increasing event A , a sequence of measures \mathbb{P}_p , and consider $g(p) = \mathbb{P}_p(A)$.

For example, consider a problem from reliability theory - an electrical network has every link cut with probability $1 - p$, and what is the probability that the network is still connected? This class of theorems are called "S-shaped theorems".

$$\Omega = \{0, 1\}^E, |E| < \infty, |E| = N, A \subseteq \Omega.$$

Let $e \in E$.

Definition 1.22. The influence of e on A is

$$I_A(e) = \mathbb{P}_p(\mathbb{I}(A)(\omega^e) \neq \mathbb{I}_A(\omega_e)). \quad (1.25)$$

If A is increasing, then

$$I_A(e) = \mathbb{P}_p(A^e) - \mathbb{P}_p(A_e). \quad (1.26)$$

where

$$A^e = \{\omega : \omega^e \in A\} \quad (1.27)$$

$$A_e = \{\omega : \omega_e \in A\} \quad (1.28)$$

$$(1.29)$$

Theorem 1.23 (Kahn-Kalani-Limial, Talagrand). *There exists $c > 0$ such that for all ϵ, A and $0 < p < 1$. Then*

$$\sum_{e \in E} I_A(e) \geq c[\mathbb{P}_p(A)\mathbb{P}_p(\bar{A})] \log \frac{1}{\max_{e \in E} I_A(e)}. \quad (1.30)$$

Proof. One uses discrete Fourier analysis (but non-examinable). \square

Theorem 1.24. *It is interesting if we have uniform upper bound M_p for the $I_A(e)$.*

Let $m = \max_{e \in E} I_A(e)$. Then we can write

$$mN \geq [\dots] \log \frac{1}{m} \quad (1.31)$$

$$m \geq \frac{[\dots]}{N} \log \frac{1}{m} \geq [\dots]' \frac{\log N}{N}. \quad (1.32)$$

Theorem 1.25 (Restatement of KKL). *The maximum influence M satisfies*

$$m \geq c' \mathbb{P}_p(A)\mathbb{P}_p(\bar{A}) \frac{\log N}{N} \quad (1.33)$$

for some universal $c' > 0$.

The $\frac{\log N}{N}$ is optimal.

Example 1.26 (Tribes). *Consider N people partitioned into t tribes, each of size $s = \log N - \log \log N + \alpha$, and let $p = \frac{1}{2}$.*

Then let

$$A = \{\text{There exists a tribe all of whose elements are } 1\} \quad (1.34)$$

Then

$$I_A(e) \sim c \mathbb{P}(A) \mathbb{P}(\bar{A}) \frac{\log N}{N} \quad (1.35)$$

for all e .

Theorem 1.27 (Symmetric Case). *If $I_A(e)$ is a constant for $e \in E$,*

$$\sum_{e \in E} I_A(e) \geq c[\mathbb{P}_p(A)\mathbb{P}_p(\bar{A})] \log N. \quad (1.36)$$

1.6 Sharp Threshold

Let Ω as before, $A \subseteq \Omega$. Then

Theorem 1.28 (Rousseau, Margoulis).

$$\frac{d}{dp} \mathbb{P}_p(A) = \sum_{e \in E} \mathbb{P}_p(A^e) - \mathbb{P}_p(A^e). \quad (1.37)$$

Note this is equal to $\sum_{e \in E} I_A(e)$ if A is increasing.

Proof. Need to only consider

$$\mathbb{P}_p(A) = \sum_{\omega} \mathbb{I}(A)(\omega) p^{|\eta|} (1-p)^{N-|\eta|} \quad (1.38)$$

where $N = |E|$, $\eta = \{e : \omega(e) = 1\}$.

Then

$$\frac{d}{dp} \mathbb{P}_p(A) = \sum_{\omega} \mathbb{I}(A)(\omega) \left(\frac{|\eta|}{p} - \frac{N-|\eta|}{1-p} \right) p^{|\eta|} (1-p)^{N-|\eta|} \quad (1.39)$$

and so

$$p(1-p) \frac{d}{dp} \mathbb{P}_p(A) = \sum_{\omega} \mathbb{I}(A)(\omega) (|\eta| - Np) p^{|\eta|} (1-p)^{N-|\eta|} \quad (1.40)$$

$$= \mathbb{P}_p(\mathbb{I}(A)(|\eta| - Np)) \quad (1.41)$$

$$= \sum_e \mathbb{P}_p(\mathbb{I}(A)(\mathbb{I}(e) - p)) \quad (1.42)$$

$$= \sum_e \mathbb{P}_p(\mathbb{I}(A)\mathbb{I}(e)) - p\mathbb{P}_p(A) \quad (1.43)$$

$$= \sum_e p\mathbb{P}_p(A^e) - p(p\mathbb{P}_p(A^e) + (1-p)\mathbb{P}_p(A_e)) \quad (1.44)$$

where $\mathbb{I}(e) = \mathbb{I}(e \text{ open}) = \omega(e)$, so $|\eta| = \sum_e \mathbb{I}(e)$.

This completes the proof. \square

1.7 Back to Percolation

Let $L^d = (\mathbb{Z}^d, \mathbb{E}^d)$, $0 < p < 1$, and measure \mathbb{P}_p . Let N be the number of open clusters. Then

$$\mathbb{P}_p(N \geq 1) = \begin{cases} 0 & p < p_c \\ 1 & p > p_c \end{cases} \quad (1.45)$$

Then $\theta(p) = \mathbb{P}_p(0 \in \text{infinite open cluster})$. So

$$\theta = \begin{cases} 0 & p < p_c \\ > 0 & p > p_c \end{cases} \quad (1.46)$$

To show (??) implies (??), we have $\mathbb{P}_p(N \geq 1) \leq \sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(x \in \text{infinite open cluster}) = \sum_x 0 = 0$.

To show (??) implies (??), by Kolmogorov's zero-one law, we have $\mathbb{P}_p(N \geq 1) \in \{0, 1\}$, but $\mathbb{P}_p(N \geq 1) \geq \theta(p) > 0$ for $p > p_c$.

Theorem 1.29 (Uniqueness of infinite cluster). *For all $0 < p < 1$, either*

$$\mathbb{P}_p(N = 0) = 1 \quad (1.47)$$

or

$$\mathbb{P}_p(N = 1) = 1 \quad (1.48)$$

Proof. Fix $p \in (0, 1)$.

Lemma 1.30 (Part A). *There exists $k = k_p \in \{0, 1, 2, \dots\} \cup \{\infty\}$ with $\mathbb{P}_p(N = k) = 1$.*

Proof. \mathbb{L}^d comes equipped with a shift translation, and the measure is invariant under this shift. Thus $N = N(\omega)$ is invariant under the shift. □

This proof requires this lemma.

Lemma 1.31. *Any shift-invariant random variable on $(\Omega, \mathcal{F}, \mathbb{P}_p)$ is almost surely constant.*

Proof. Elementary application of measure theory. □

Lemma 1.32 (Part B). $k_p \in \{0, 1, \infty\}$ - the "finite-energy property".

Proof. Suppose $2 \leq k_p < \infty$.

Find n such that $\mathbb{P}_p(\Lambda_n \text{intersections} \geq 2 \text{ infinite open clusters}) > \frac{1}{2}$. □

Lemma 1.33 (Part C). $k_p \neq \infty$.

Proof. Say x is a trifurcation if

Follow this argument? p93 in
Probability on Graphs

- (i) $|C_x| = \infty$.
- (ii) The removal of x breaks C_x into three disjoint infinite clusters.

Then $\tau = \mathbb{P}_p(x \text{ is a trifurcation})$ is independent of x .

We claim $\tau > 0$. To show this, take a large diamond box that intersects with at least three open clusters. Then there exists n such that $\mathbb{P}_p(S_n \text{ intersects } \geq 3 \text{ infinite open clusters}) > \frac{1}{2}$.

Thus $\tau > 0$.

The argument is then that we use the ratio between boundary and volume to bound the number of trifurcations in A_n , and show that this leads to a contradiction for large n . \square

N.B. - consider the corresponding proof for site percolation. For $x, y, z \in \partial S$, does there exist open paths to zero? \square

Fill in rest of proof. Required to understand high-level ideas and key steps around the graphs

1.8 Percolation in Two Dimensions

There are two models, bond percolation on L^2 , and site percolation on Π , the triangular lattice.

The triangular lattice is "self-matching", in that the dual construction is on the same lattice as the primal (c.f. the dual of the square lattice).

1.8.1 Bond percolation on \mathbb{Z}^2 , Site Percolation on \mathbb{T}

Theorem 1.34. For bond percolation on \mathbb{Z}^2 , $\Theta(\frac{1}{2}) = 0$.

Proof (Proof of Zhang). Let $p = \frac{1}{2}$ and suppose $\Theta(\frac{1}{2}) > 0$. Since $\Theta(\frac{1}{2}) > 0$, then the probability there exists an infinite open cluster is one.

Let $T_n = [0, n]^2$. As n goes to infinity, then the probability that T_n intersects with the infinite open cluster tends to one. Thus, find N such that for all $n > N$, $\mathbb{P}(T_n \text{ intersects the infinite open cluster})$ is greater than $1 - \frac{1}{8}$.

Consider A^t be the event that the **top** of T_n is joined to the infinite open cluster. Define A^b, A^l, A^r to be the **bottom, left, and**

right analogues. Then $\mathbb{P}(T_n \text{ does not intersect the infinite cluster})$

is $\mathbb{P}(\overline{A^t} \cap \overline{A^b} \cap \overline{A^l} \cap \overline{A^r}) \geq \mathbb{P}(\overline{A^u})^4$ for $u = t, b, l, r$.

Then we have $\mathbb{P}(A^u) \geq \frac{7}{8}$ by the given result.

Let $n = N + 1$. Pass to the dual percolation, ...

□

Fill this in from the Probability on Graphs book. Doesn't look too difficult.

1.8.2 Site percolation on Π

Π has the vertex set $\{m\tilde{i} + n\tilde{j} : m, n \in \mathbb{Z}\}$, $\tilde{i} = (1, 0)$, $\tilde{j} = \frac{1}{2}(1, \sqrt{3})$ when embedded into \mathbb{R}^2 .

Now, consider a box in \mathbb{R}^2 , with vertices $(0, 0)$ and (a, b) with $a \in \mathbb{N}$, $b \in \sqrt{3}2\mathbb{N}$.

Each site is black with probability $\frac{1}{2}$, and white otherwise. Let $H_{a,b} = \{L \leftrightarrow^{black} R \in R_{a,b} \text{ where } L \text{ is the left edge and } R \text{ is the right edge. That is, } H_{a,b} \text{ is the event that there exists a black path that traverses } R_{a,b} \text{ from } L(R_{a,b}) \text{ to } R(R_{a,b}).$

Then we have the lemma as follows:

Lemma 1.35 (RSW Lemma).

$$\mathbb{P}(H_{2a,b}) \geq \frac{1}{4} \mathbb{P}(H_{a,b})^2. \tag{1.49}$$

Proof on p100-105 in book

Theorem 1.36. $p_c(\text{bond}, \mathbb{T}) = \frac{1}{2}$

Theorem 1.37. $p_c \geq \frac{1}{2}$, and in fact $\Theta(\frac{1}{2}) = 0$ for the bond model on \mathbb{Z}^2 .

Proof. Following p 122 of the book.

We need to prove that $p_c \leq \frac{1}{2}$ - that is $\Theta(p) > 0$ for $p > \frac{1}{2}$.

Let $H_n = H_{16n, n\sqrt{3}}$ be the event that a black crossing of $R_{16n, n\sqrt{3}}$ exists. By the previous lemma, there exists $\tau > 0$ such that $\mathbb{P}_{\frac{1}{2}}(H_n) \geq \tau$ for some $\tau > 0$. Let $\frac{1}{2} \leq p \leq \frac{3}{4}$.

Then

$$(1 - p)I_{n,p}(x) \leq \mathbb{P}_{1-p}(\text{Rad}(C_x) \geq n) \leq \mathbb{P}_{\frac{1}{2}}(\text{Rad}(C_0) \geq n) = v_n \rightarrow 0 \tag{1.50}$$

where $\text{Rad}(C_x) = \max\{|y - x| : x \leftrightarrow y\}$.

So we have

$$\frac{d}{dp} \mathbb{P}_p(H_n) \geq c\tau(1 - \mathbb{P}_p(H_n)) \log \frac{1}{8\nu_n} \quad (1.51)$$

and integrating gives

$$\int_{\frac{1}{2}}^p \frac{g'(p)}{1 - g(p)} dp \geq c\tau \log \frac{1}{8\nu_n} \left(p - \frac{1}{2}\right) \quad (1.52)$$

and so $\mathbb{P}_p(H_n) \geq 1 - (1 - \tau)(8\nu_n)^{c\tau(p - \frac{1}{2})} \rightarrow 1$ as $n \rightarrow \infty$ if $p > \frac{1}{2}$.

... □

Fill in rest of proof (block argument)

1.9 Cardy's Formula

Given a Jordan curve on R^2 , there exists a conformal map from D to the interior of the equilateral triangle T of C with vertices $A = 0, B = 1, C = e^{\frac{\pi i}{3}}$ and such that ϕ can be extended to the boundary ∂D in such a way that it becomes a homeomorphism from $D \cup \partial D$ to the closed triangle T .

Theorem 1.38 (Cardy's Formula).

$$\mathbb{P}_\delta(ac \leftrightarrow bx \text{ in } D) \rightarrow |BX| \quad (1.53)$$

as $\delta \rightarrow 0$.

Fill in the rest of the proof of Cardy's formula.

2

Self Avoiding Walks

Consider G a graph, with γ is a self avoiding walk which visits each vertex of G at most once.

$G_n(v)$ is the number of self avoiding walks with length n . We assume G is transitive. Then G_n is submultiplicative, and we defined

$$K(G) = \lim_{n \rightarrow \infty} \sqrt[n]{G_n} \quad (2.1)$$

For examples $K(d - \text{ary} - \text{tree}) = d - 1$, and $K(G) \leq \Delta - 1$ (exercise).

Theorem 2.1. $K(H1) = \sqrt{2 + \sqrt{2}}$.

2.1 Generating Functions

$$Z(z) = \sum_{\gamma \text{ SAW}} \sum_{n=1}^{\infty} G_n \cdot z^n \quad (2.2)$$

Cauchy-Hadamard gives the radius of convergence is $\frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{G_n}} = \frac{1}{K(G)}$

Fill in proof of $\sqrt{2 + \sqrt{2}}$

2.2 Random Clusters Model/FK (Fortun-Kostelyn) Percolation

Definition 2.2. $G = (V, E)$, $\Omega = \{0, 1\}^E$, for $\omega \in \Omega$, $k(\omega)$ is the number of open clusters.

The RC measure $p \in [0, 1], q \in (0, \infty)$,

$$\phi_{p,q}(\omega) = \frac{1}{Z_{p,q}} \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} q^{k(\omega)} \quad (2.3)$$

- (i) $q = 1$ is standard percolation,
- (ii) $p, q \rightarrow 0$ with $\frac{q}{p} \rightarrow 0$ is electrical networks.
- (iii) For $q = 2$, we have the FK Ising model, where for $\omega \in \{0, 1\}^E$, for each open cluster of ω , we set the spins/states of the vertices of it to ± 1 iwht equal probability, so $G \in \{\pm 1\}^V$, with

$$\mu_\beta(G) = \frac{1}{Z_\beta} \exp(\beta \sum_{x \sim y} G_x G_y) \quad (2.4)$$

and $p = 1 - e^{-\beta}$.

Our aim is to define the random cluser measure on (\mathbb{Z}^d, E^d) . Let Λ be a finite box in \mathbb{Z}^d . Let $E_\Lambda = \{(u, v) \in E^d \mid u, v \in \Lambda\}$. Let $b = 0, 1$, and $\Omega_\Lambda^b = \{\omega \in \Omega = \{0, 1\}^{E^d} \mid \omega(e) = b \forall e \notin \Lambda\}$.

Let $\phi_{\Lambda,p,q}^b(\omega) = \frac{1}{Z_{\Lambda,p,q}} (\prod_{e \in E_\Lambda} p^{\omega(e)} (1-p)^{1-\omega(e)}) q^{k(\omega, \Lambda)}$.

3

Bibliography