CONVEX OPTIMIZATION SUMMARY

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1. Existence

Definition. *For $C \subseteq \mathbb{R}^n$, define δ_C as*

$$\delta_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases} \tag{1.1}$$

Note x' minimizes f over C if and only if x' minimizes $f + \delta_C$ over $\mathbb{R}^n$.

Definition. *(i)* $\operatorname{dom} f = \{x \in \mathbb{R}^n | f(x) < \infty\}$,

(ii)

$$\arg\min f = \begin{cases} \emptyset & f \equiv \infty \\ \{x \in \mathbb{R}^n | f(x) = \inf f\} & f < \infty \end{cases} \tag{1.2}$$

*(iii) f is **proper** if and only if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$.*

Definition. *For $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, denote*

$$\operatorname{epi} f = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} | f(x) \leq \alpha\} \tag{1.3}$$

Theorem. A set C is an epigraph if and only if for every x there is an $\alpha \in \overline{\mathbb{R}}$ such that $C \cap (x \times \mathbb{R}) = [\alpha, \infty]$ - so all vertical one-dimensional sections must be closed upper half-lines. If f is proper then $\operatorname{epi} f$ is not empty and does not include a complete vertical line.

Definition. *For $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, define*

$$\liminf_{x \to x'} f(x) = \lim_{\delta \downarrow 0} \inf_{\|x - x'\|_2 \leq \delta} f(x) = \lim_{k \to \infty} \inf_{\|x - x'\|_2 \leq \frac{1}{k}} f(x) \tag{1.4}$$

*f is **lower semicontinuous** at x' if and only if $f(x') \leq \liminf_{x \to x'} f(x)$. f is **lower semicontinuous** if f is lower semicontinuous at every $x' \in \mathbb{R}^n$.*

Theorem.

$$\liminf_{x \to x'} f(x) = \min\{\alpha \in \overline{\mathbb{R}} | \exists (x^k) \to x' : f(x^k) \to \alpha\} \tag{1.5}$$

In particular, f is lower semi-continuous at x' if and only if $f(x') \leq \liminf_{k \to \infty} f(x^k)$ for all convergence sequences $x^k \to x'$.

Theorem. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. Then the following are equivalent:

(i) f is lsc on $\mathbb{R}^n$
(ii) $\operatorname{epi} f$ is closed in $\mathbb{R}^n \times \mathbb{R}$
(iii) The sub level sets $\operatorname{lev}_{\leq \alpha} f = \{x \in \mathbb{R}^n | f(x) \leq \alpha\}$ are closed in $\mathbb{R}^n$ for all $\alpha \in \overline{\mathbb{R}}$.

Proof. $\operatorname{epi} f$ can only be not closed along vertical lines. □

Definition. *$f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is level bounded if and only if $\operatorname{lev}_{\leq \alpha} f$ is bounded for all $\alpha \in \mathbb{R}$.*

Theorem. $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is level-bounded if and only if $f(x^k) \to \infty$ for all sequences (x^k) satisfying $\|x^k\|_2 \to \infty$.

Proof. $K(\alpha)$ such that $x^k \notin \operatorname{lev}_{\leq \alpha} f$ for $k \geq K(\alpha)$ by boundedness. In reverse, find α with $\operatorname{lev}_{\leq \alpha} f$ unbounded and choose x^k in this set with $f(x^k) \leq \alpha$. □

Theorem. Assume $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is lsc, level-bounded, and proper. Then $\inf f(x) \in (-\infty, +\infty)$ and $\arg\min f$ is nonempty and compact.

Proof. Consider $\cap_{\alpha \in \mathbb{R}, a > \in f} = \arg\min f$, then this is countable intersection of nonempty, compact sets, so intersection is nonempty.

Finiteness of inf is shown as for any $x \in \arg\min f \neq \emptyset$, must have $f(x) = -\infty$ contradicting properness. □

Theorem. (i) f, g lsc, proper implies $f + g$ is lsc.
(ii) f lsc, $\lambda \geq 0$ implies λf is lsc
(iii) $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is lsc and $g : \mathbb{R}^m \to \mathbb{R}^n$ is continuous implies $f \circ g$ is lsc.

2. Convexity

Definition. *$f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex if and only if*

$$f((1 - \tau)x + \tau y) \leq (1 - \tau) f(x) + \tau f(y) \tag{2.1}$$

for all $x, y \in \mathbb{R}^n, \tau \in (0, 1)$.

A set $C \subseteq \mathbb{R}^n$ is convex if and only if δ_C is convex if and only if $(1 - \tau)x + \tau y \in C$ for all $x, y \in C, \tau \in (0, 1)$.

$f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is strictly convex if and only if f is convex and the inequality is strict for all $x \neq y$ and $\tau \in (0, 1)$.

Definition. *Let $x_0, \dots, x_m \in \mathbb{R}^n$ and $\lambda_0, \dots, \lambda_m \geq 0$, $\sum_{i=0}^m \lambda_i = 1$. The linear combination $\sum_{i=0}^m \lambda_i x_i$ is a **convex combination** of the points $x_0, \dots, x_m$.*

Theorem. (i) $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex if and only if

$$f\left(\sum_{i=0}^m \lambda_i x_i\right) \leq \sum_{i=0}^m \lambda_i f(x_i) \tag{2.2}$$

for all $m \geq 0, x_i \in \mathbb{R}^n, \lambda_i \geq 0, \sum_{i=0}^m \lambda_i = 1$.
(ii) $C \subseteq \mathbb{R}^n$ is convex if and only if C contains all convex combinations of its elements.

Theorem. $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex implies $\operatorname{dom} f$ is convex.

Theorem. (i) $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex if and only if $\operatorname{epi} f$ is convex in $\mathbb{R}^n \times \mathbb{R}$.
(ii) $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is strictly convex if and only if $\{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} | f(x) < \alpha\}$ is convex in $\mathbb{R}^n \times \mathbb{R}$.
(iii) $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex implies $\operatorname{lev}_{\leq \alpha} f$ is convex for all $\alpha \in \overline{\mathbb{R}}$.

Theorem. Assume $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex. Then

(i) $\arg\min f$ is convex.
(ii) x is a local minimizer of f implies x is a global minimize of f.
(iii) f is strictly convex and proper implies f has at most one global minimizer.

Theorem. Let I be an arbitrary index set. Then

(i) $f_i, i \in I$ convex implies $\sup_{i \in I} f_i(x)$ is convex.
(ii) $f_i, i \in I$ strictly convex, I finite implies $\sup_{i \in I} f_i(x)$ is strictly convex.
(iii) $C_i, i \in I$ is convex implies $\cap_{i \in I} C_i$ is convex.
(iv) $f_k, k \in N$ is convex implies $\limsup_{k \to \infty} f_k(x)$ is convex.

Theorem. Assume $C \subseteq \mathbb{R}^n$ is open and convex, and $f : C \to \mathbb{R}$ is differentiable. Then the following are equivalent:

(i) f is [strictly] convex
(ii) $\langle y - x, \nabla f(y) - \nabla f(x) \rangle \geq 0$ for all $x, y \in C$ [and > 0 if $x \neq y$]
(iii) $f(x) + \langle y - x, \nabla f(x) \rangle \leq f(y)$ for all $x, y \in C$ [and $< f(y)$ if $x \neq y$]
(iv) If f is additionally twice differentiable, then $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$.

Proof. Reduce to one-dimensional sections $g(t) = f(x + t(x - y))$. Take $g(x + h)$, use convexity to show $g'(x) \leq \frac{g(y) - g(x)}{y - x}$.

Use $g(x) = \sup_{y \in \operatorname{dom} g} g(y) + (y - x) g'(y)$ to show the supremum over convex functions is convex. □

Theorem. (i) Assume $f_1, \dots, f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ are convex, $\lambda_1, \dots, \lambda_m \geq 0$. Then $f = \sum_{i=1}^m \lambda_i f_i$ is convex. If at least one of the f_i with $\lambda_i > 0$ is strictly convex, then f is strictly convex.
(ii) Assume $f_i : \mathbb{R}^{n_i} \to \overline{\mathbb{R}}$ are convex. Then

$$f : \mathbb{R}^{n_1} \times \dots R^{n_m} \to \overline{\mathbb{R}} f(x_1, \dots, x_m) = \sum_{i=1}^m f_i(x_i) \tag{2.3}$$

is convex. If all f_i are strictly convex, then f is strictly convex.
(iii) If $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ is convex, $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$. Then $g(x) = f(Ax + b)$ is convex.

Theorem. (i) $C_1, \dots, C_m$ convex implies $C_1 \times \dots \times C_m$ is convex.
(ii) $C \subseteq \mathbb{R}^n$ convex, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $L(x) = Ax + b$ implies $L(C)$ is convex.
(iii) $C \subseteq \mathbb{R}^m$ convex, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $L(x) = Ax + b$ implies $L^{-1}(C)$ is convex.
(iv) C_1, C_2 is convex implies $C_1 + C_2$ is convex.
(v) C convex, $\lambda \in \mathbb{R}$ implies λC is convex.

Definition. *For a set $S \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, define the projection of x onto S as*

$$\Pi_S(y) = \arg\min_{x \in S} \|x - y\|_2. \tag{2.4}$$

Theorem. Assume $C \subseteq \mathbb{R}^n$ is convex, closed, and $C \neq \emptyset$. Then Π_C is single-valued - that is, the projection of x onto C is unique for every $x \in \mathbb{R}^n$.

Proof. f is lsc, level-bounded, and proper, so $\arg\min f \neq \emptyset$. f is strictly convex so has at most one minimizer. □

Definition. *For an arbitrary set $S \subseteq \mathbb{R}^n$,*

$$\operatorname{con} S = \bigcap_{C \text{ convex, } S \subseteq C} C \tag{2.5}$$

is the convex hull of S. It is the smallest convex set that contains S.

Theorem. $\operatorname{con} S = \{\sum_{i=0}^p \lambda_i x_i | x_i \in S, \lambda_i \geq 0, \sum_{i=0}^p \lambda_i = 1, p \geq 0\} = D$

Proof. $D \subseteq \operatorname{con} S$ as a convex set contains all convex combinations of points in S, thus $D \subseteq \operatorname{con} S$.

$\operatorname{con} S \subseteq D$ as taking linear combination of $x, y \in D$, we have a convex combination of elements in D, which is in D, so $\operatorname{con} S \subseteq D$. □

Theorem. We have

(i) $\operatorname{cl} C = \{x \in \mathbb{R}^n | \forall$ open neighborhoods N of x, $N \cap C \neq \emptyset\}$
(ii) $\operatorname{int} C = \{x \in \mathbb{R}^n | \exists$ open neighborhood N of x such that $N \subseteq C\}$
(iii) $\operatorname{bnd} C = \operatorname{cl} C \setminus \int C$.

Theorem. $\operatorname{cl} C = \bigcap_{S \text{ closed, } C \subseteq S} S$.

3. Cones and Generalized Inequalities

Definition. *$K \subseteq \mathbb{R}^n$ is a cone if and only if $0 \in K$ and $\lambda x \in K$ for all $x \in K, \lambda \geq 0$.*

A cone K is pointed if and only if $\sum_{0=1}^m x_i = 0$, $x_i \in K$ implies $x_i = 0$ for all i.

Theorem. Let $K \subseteq \mathbb{R}^n$ be arbitrary. Then the following are equivalent:

(i) K is a convex cone.
(ii) K is a cone and $K + K \subseteq K$.
(iii) $K \neq \emptyset$ and $\sum_{i=0}^m \alpha_i x_i \in K$ for all $x_i \in K$ and $\alpha_i \geq 0$ (not necessarily summing to 1).

Proof. $x = \sum_{i=0}^m \alpha_i x_i \in K$, $a_i \geq 0 \iff \sum_{i=0}^m \frac{a_i}{\sum_j \alpha_j} x \in K$, which is a convex combination. □

Theorem. Assume K is a convex cone. Then K is pointed if and only if $K \cap -K = \{0\}$.

Proof. $x_1 + x_2 + \cdots + x_m \in K$, $x_1 \neq 0$, $x_2 + \cdots + x_m = -x_1$, $x_2 + \cdots + x_m \in K$, so $x_1 \in K \cap -K$. □

Theorem. For a closed convex cone $K \subseteq \mathbb{R}^n$ we define the generalized inequality

$$x \leq_K y \iff x - y \in K \tag{3.1}$$

Then

(i) $x \leq_K x$
(ii) $x \leq_K y, y \leq_K z \Rightarrow x \leq_K z$
(iii) $x \leq_K y \Rightarrow -y \leq_K -x$
(iv) $x \leq_K y, \lambda \geq 0 \Rightarrow \lambda x \geq_K \lambda y$
(v) $x \geq_K y, x' \geq_K y' \Rightarrow x + x' \geq_K y + y'$
(vi) $x^k \to x$, $y^k \to y$ with $x^k \geq_K y^K$ for all $k \in \mathbb{N}$, then $x \geq_K y$.
(vii) $x \geq_K y, y \geq_K \Rightarrow x = y$ (antisymmetry) holds if and only if K is pointed.

Definition (K_n^{LP})**.** *For any pointed, closed, convex cone $K \subseteq \mathbb{R}^m$, a matrix $A \in \mathbb{R}^{m \times n}$, and vectors $c \in \mathbb{R}^n, b \in \mathbb{R}^m$, define the conic problem* $\inf_x c^T x s.t. Ax \geq_K b$.

Definition (K_n^{SOCP})**.**

$$K_n^{SOCP} = \{x \in \mathbb{R}^n\} | x_n \geq \sqrt{x_1^2 + \cdots + x_{n-1}^2}. \tag{3.2}$$

4. Subgradients

Definition. *For any $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $x \in \mathbb{R}^n$,*

$$\partial f(x) = \{v \in \mathbb{R}^n | f(x) + \langle v, y - x \rangle \leq f(y) \forall y \in \mathbb{R}^n\} \tag{4.1}$$

is the set of subgradients of f at x.

Theorem. Assume $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ are convex. Then

(i) If f is differentiable at x, then $\partial f(x) = \{\nabla f(x)\}$
(ii) If f is differentiable at x and $g(x) \in \mathbb{R}$, then $\partial(f + g)(x) = \partial g(x) + \nabla f(x)$.

Theorem. Assume $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is proper. Then $x \in \arg\min f \iff 0 \in \partial f(x)$.

Proof.

$$0 \in \partial f(x) \iff f(x) \leq f(y) \forall y \iff x \in \arg\min f \tag{4.2}$$

where properness was required since $\arg\min +\infty = \emptyset$ by definition. □

Definition. *For a convex set $C \subseteq \mathbb{R}^n$ and $x \in C$, the normal cone $N_C(x)$ at x is*

$$N_C(x = \{v \in \mathbb{R}^n | \langle v, y - x \rangle \leq 0 \forall y \in C\}) \tag{4.3}$$

$N_C(x) = \emptyset$ for $x \notin C$.

Note that $N_C(x)$ is a cone if $x \in C$.

Theorem. Assume $C \subseteq \mathbb{R}^n$ is convex with $C \neq \emptyset$. Then $\partial \delta_C(x) = N_C(x)$.

Proof. For $x \in C$, this follows easily, and for $x \notin C$, choosing $y \in C$ shows that $\partial \delta_C(x) = \emptyset$. □

Theorem. Assume $C \subseteq \mathbb{R}^n$ is closed and convex with $C \neq \emptyset$ and $x \in \mathbb{R}^n$. Then $y \in \Pi_C(x) \iff x - y \in N_C(y)$.

Proof. Follows from y being the unique minimizer of $f(y') = \frac{1}{2}\|y' - x\|_2^2 + \delta_C(y')$. □

Theorem. Assume $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is proper and convex. Then

$$\partial f(x) = \begin{cases} \emptyset & x \notin \operatorname{dom} f \\ \{v \in \mathbb{R}^n | (v, -1) \in N_{\operatorname{epi} f}(x, f(x))\} & x \in \operatorname{dom} f \end{cases} \tag{4.4}$$

If $x \in \operatorname{dom} f$ then $N_{\operatorname{dom} f}(x) = \{v \in \mathbb{R}^n | (v, 0) \in N_{\operatorname{epi} f}(x, f(x))\}$.

Proof. $x \notin \operatorname{dom} f$: choose $f(y) < \infty$, then $\partial f(x) = \emptyset$.

$x \in \operatorname{dom} f$: $v^T(y - x) + (-1)(-f(x)) \leq 0 \forall (y, \alpha) \in \operatorname{epi} f \iff (v, -1) \in N_{\operatorname{epi} f}(x, f(x))$. □

Definition. *For any set $C \subseteq \mathbb{R}^n$, define the affine hull and relative interior by*

$$\operatorname{aff} C = \cap_{A \text{ affine, } C \subseteq A} A \tag{4.5}$$

$$\operatorname{rint} C = \{x \in \mathbb{R}^n | \exists \text{ open neighborhood } N \text{ of } x \text{ with } N \cap \operatorname{aff} C \subseteq C\}. \tag{4.6}$$

Theorem. (i) Assume $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex. Then

$$g(x) = f(x + y) \Rightarrow \partial g(x) = \partial f(x + y) \tag{4.7}$$

$$g(x) = f(\lambda x) \Rightarrow \partial g(x) = \lambda \partial f(\lambda x), \lambda \neq 0 \tag{4.8}$$

$$g(x) = \lambda f(x) \Rightarrow \partial g(x) = \lambda \partial f(x), \lambda > 0 \tag{4.9}$$

(4.10)

(ii) Assume $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is proper and convex, and $A \in \mathbb{R}^{n \times m}$ is such that

$$\{Ay | y \in \mathbb{R}^m\} \cap \operatorname{rint} \operatorname{dom} f \tag{4.11}$$

If $x \in \operatorname{dom}(f \circ A) = \{y \in \mathbb{R}^m | Ay \in \operatorname{dom} f\}$, then $\partial(f \circ A)(x) = A^T \partial f(Ax)$.

(iii) Assume $f_0, \ldots, f_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ are proper and convex, and $\operatorname{rint} \operatorname{dom} f_0 \cap \cdots \cap \operatorname{rint} \operatorname{dom} f_m \neq \emptyset$. If $x \in \operatorname{dom} f$, then $\partial(\sum_{i=0}^m f_i)(x) = \sum_{i=0}^m \partial f_i(x)$.

5. Conjugate Functions

Definition. *For $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\operatorname{con} f(x) = \sup_{g \leq f, \, g \text{ convex}}$ is the convex hull of f. $\operatorname{con} f$ is the greatest convex function majorized by f.*

Definition. *For $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, the* ***lower closure*** *$\operatorname{cl} f$ is defined as $(\operatorname{cl} f)(x) = \liminf_{y \to x} f(y)$. Alternatively, $(\operatorname{cl} f)(x) = \sup_{g \leq f, \, g \text{ lsc}} g(x)$.*

Theorem. For $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we have $\operatorname{epi}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{epi} f)$. Moreover, if f is convex then $\operatorname{cl} f$ is convex.

Proof. $(x, \alpha) \in \operatorname{cl}(\operatorname{epi} f) \iff \exists x^k \to x, \alpha^k \to \alpha, f(x^k \leq \alpha^k) \iff \liminf_{y \to x} f(y) \leq \alpha \iff (x, \alpha) \in \operatorname{epi}(\operatorname{cl} f)$. □

Theorem. Assume $C \subseteq \mathbb{R}^n$ is closed and convex. Then

$$C = \bigcap_{(b,\beta), C \subseteq H_{b,\beta}} H_{b,\beta} \tag{5.1}$$

where $B_{b,\beta} = \{x \in \mathbb{R}^n | \langle x, b\rangle - \beta \leq 0\}$

Proof. Separating hyperplane theorem - $x \in C$ then x is the intersection. If $x \notin C$ consider the separating hyperplane of C and $\{x\}$. ☐

Theorem. Assume $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is proper, lsc, and convex. Then $f(x) = \sup_{g \text{ affine, } f \leq f} g(x)$.

Proof. f lsc implies $\operatorname{epi} f$ is closed, f is convex implies $\operatorname{epi} f$ is convex, so epi f is the intersection of all half spaces containing it. Show g affine $\iff$ there exists $(b, c), \beta$ with $c < -$, $\operatorname{epi} g = H_{(b,c),\beta}$.

Then as $\operatorname{epi}(\sup_{g \leq f} g(x)) = \cap_{g \leq f} \operatorname{epi} g$ where g is affine and $g \leq f \iff \operatorname{epi} f \subseteq \operatorname{epi} g$, we need only show $I_1 = \cap_{(b,c),\beta \in S} H_{(b,c),\beta} = \cap_{(b,c),\beta \in S, c<0} H_{(b,c),\beta} = I_2$. ☐

Definition. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, then*

$$f^\star : \mathbb{R}^n \to \overline{\mathbb{R}} \tag{5.2}$$

$$f^\star(v) = \sup_{x \in \mathbb{R}^n} \langle v, x\rangle - f(x) \tag{5.3}$$

is the ***conjugate to*** *f. The mapping $f \mapsto f^\star$ is the Legendre-Fenchel transform.*

Theorem. Assume $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. Then $f^\star = (\operatorname{con} f)^\star = (\operatorname{cl} f)^\star = (\operatorname{cl} \operatorname{con} f)^\star$ and $f^{\star\star} = (f^\star)^\star \leq f$. If $\operatorname{con} f$ is proper, then $f^\star$ and $f^{\star\star}$ are proper, lsc, and convex, and $f^{\star\star} = \operatorname{cl} \operatorname{con} f$. If $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is proper, lsc, and convex, then $f^{\star\star} = f$.

Proof. $(v, \beta) \in \operatorname{epi} f^\star \iff (v, x) - \beta \leq f(x) \forall x$ so $(v, \beta) \in \operatorname{epi} f\star$ define all affine functions majorized by f. Thus for every affine function h, $h \leq f \iff h \leq \operatorname{cl} f \iff h \leq \operatorname{con} f \iff h \leq \operatorname{cl} \operatorname{con} f$. Which gives our result.

For the inequality, expand $f^{\star\star}(y) = \sup_v \langle v, y\rangle + \inf_x (f(x) - \langle v, x\rangle) \leq f(y)$ at $y = x$.

As $\operatorname{con} f$ is proper, then $\operatorname{cl} \operatorname{con} f$ is proper, lsc, and convex, so $\operatorname{cl} \operatorname{con} f = \sup_{g \leq \operatorname{cl} \operatorname{con} f} g(x)$ where g is affine,which equals $f^{\star\star}$ by definition.

Finally, if f is convex, then $\operatorname{con} f = f$, so $\operatorname{con} f$is proper, and $\operatorname{con} f = f$ is lsc, so $f^{\star\star} = \operatorname{cl} \operatorname{con} f = \operatorname{cl} f = f$. ☐

Theorem. Assume $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. Then

(i) $\operatorname{con} f$ is not proper implies $f^\star \equiv +\infty$ or $f^\star \equiv -\infty$.
(ii) In particular, $f^\star$ proper implies $\operatorname{con} f$ is proper.

Proof. If $\operatorname{con} f = \infty$, then $f^\star(v) = -\infty$. If $\operatorname{con} f(x') = -\infty$, then $f^\star(v) = (\operatorname{con} f)^\star(v) \geq +\infty$. ☐

Theorem. Assume $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is proper, lsc, and convex. Then $\partial f^\star = (\partial f)^{-1}$, specifically,

$$v \in \partial f(x) \iff f(x) + f^\star(v) = \langle v, x\rangle \iff x \in \partial f^\star(v). \tag{5.4}$$

Moreover,

$$\partial f(x) = \arg\max_{v'} \langle v', x\rangle - f^\star(v') \partial f^\star(x) = \arg\max_{x'} \langle v, x'\rangle - f(x) \tag{5.5}$$

Proof. $f(x) + f^\star(v) = \langle v, x\rangle$ iff $x \in \arg\max_{x'} \langle v, x'\rangle - f(x') \iff v \in \partial f(x)$. ☐

Theorem. For a proper, lsc, convex $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we have

$$(f(\cdot) - \langle a, \cdot\rangle)^\star = f^\star(\cdot + a) \tag{5.6}$$

$$(f(\cdot + b))^\star = f^\star(\cdot) - \langle \cdot, b\rangle, \tag{5.7}$$

$$(f(\cdot) + c)^\star = f^\star(\cdot) - c \tag{5.8}$$

$$(\lambda f(\cdot))^\star = \lambda f^{star}(\frac{\cdot}{\lambda}), \lambda > 0 \tag{5.9}$$

$$(\lambda f(\frac{\cdot}{\lambda}))^\star = \lambda f^\star(\cdot), \lambda > 0 \tag{5.10}$$

Theorem. Let $f_i : \mathbb{R}^{n_i} \to \overline{\mathbb{R}}$, $i = 0, \ldots, m$ be proper and $f(x_0, \ldots, x_m) = \sum_{i=0}^m f_i(x_i)$. Then $f^\star(v_1, \ldots, v_m) = \sum_{i=0}^m f_i^\star(v_i)$.

Proof. For support functions, $f^\star(x) = \sigma_C(x)$, so C nonempty closed convex implies $f^\star$ is proper lsc convex with $f^\star(0) = 0$ and $f^\star(\lambda v) = \lambda f^\star(v)$, so $f^\star = \sigma_C$ is positively homogeneous.

If g is positively homogeneous lower semicontinuous, $g^\star(x) \in \{0, \infty\}$, so $g^\star$ is an indicator function, which must be convex and nonempty by properness lsc convexity of $g^\star$.

One to one follows from $\delta_C^{\star\star} = \delta_C$.

For cones, show g is positively homogeneous lsc convex proper indicator function $\iff g = \delta_K$ for Ka convex closed cone.

Taking any such indicator function δ_K, then $\delta_K^\star$ is an indicator function, with $\delta_K^\star(v) < \infty \iff v \in K^\star$. ☐

Definition. *For any set $S \subseteq \mathbb{R}^n$ define the* ***support function*** $\operatorname{supp}_S(v) = \sup_{x \in S} \langle v, x\rangle = (\delta_S^\star)(v)$

Definition. *A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be positively homogeneous if and only if $0 \in \operatorname{dom} f$ and $f(\lambda x) = \lambda f(x)$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$.*

Theorem. The set of positively homogeneous proper lsc convex functions and the set of closed convex nonempty sets are in one-to-one correspondence through the Legendre-Fenchel transform:

$$\delta_C \leftrightarrow_\star \operatorname*{supp}_C \tag{5.11}$$

$$x \in \partial \operatorname*{supp}_C(v) \iff x \in C \tag{5.12}$$

$$\operatorname*{supp}_C(v) = \langle v, x\rangle \iff v \in N_C(x). \tag{5.13}$$

In particular, the set of closed convex cones is in one-to-one correspondence with itself - for any cone K define the **polar cone** or **dual cone** as $K^\star = \{v \in \mathbb{R}^d | \langle v, x\rangle \leq 0 \forall x \in K\}$. Then

$$\delta_K \leftrightarrow_\star \delta_{K^\star} \tag{5.14}$$

$$x \in N_{K^\star}(v) \iff v \in N_K(x) \tag{5.15}$$

TODO: fill next condition/implication in.

6. Duality in Optimization

Definition. *Assume $f : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ is proper, lsc, and convex. Define the* ***primal problem*** *as*

$$\inf_{x \in \mathbb{R}^n} \phi(x), \phi(x) = f(x, 0), \tag{6.1}$$

the ***dual problem*** *as*

$$\sup_{y \in \mathbb{R}^n} \psi(y), \psi(y) = -f^\star(0, y) \tag{6.2}$$

and the ***inf-projections***

$$p(u) = \inf_x f(x, u) \tag{6.3}$$

$$q(v) = \inf_y f^\star(v, y) = -\sup_y(-f^\star(v, y)) \tag{6.4}$$

f is sometimes called a ***perturbation function*** *for ψ, and p the associated* ***marginal function****.*

Theorem. Assume $f : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ is proper, lsc, and convex. Then

(i) ϕ and $-\psi$ are lsc and convex.
(ii) p, q are convex.
(iii) $p(0)$ and $p^{\star\star}(0)$ are the optimal values of the primal and dual problems -

$$p(0) = \inf_x \phi(x), p^{\star\star}(0) = \sup_y \psi(y). \tag{6.5}$$

(iv) The primal and dual problems are feasible if and only if their associated marginal function contains 0:

$$\inf_x \phi(x) < \infty \iff 0 \in \operatorname{dom} p \tag{6.6}$$

$$\sup_y \psi(y) > -\infty \iff 0 \in \operatorname{dom} q \tag{6.7}$$

Proof. f proper lsc convex implies $f^\star$ is proper lsc convex implies π, ψ lsc convex.

p, q are convex from the strict epigraph set, with $E = \{(u, a) \in \mathbb{R}^m \times \mathbb{R} | p(u) = \inf_{x \in \mathbb{R}^n} f(x, u) < \alpha\} = A(E')$, where A is the linear projection mapping $A(x, u, \alpha) = (u, \alpha)$, and E' is the strict epigraph of f and thus convex, so $A(E')$ is convex.

$p^\star(y) = -\psi(y)$, so $p^{\star\star}(0) = \sup_y \psi(y)$.

$0 \in \operatorname{dom} p \iff p(0) < \infty \iff \inf_x f(x, 0) < \infty \iff \inf \psi < \infty$. ☐

Theorem. Assume $f : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ is proper, lsc, and convex. Then **weak duality** always holds,

$$\inf_x \phi(x) \geq \sup_y \psi(y), \tag{6.8}$$

and under certain conditions the infimum and supremum are equal and finite - **strong duality**

$$p(0) \in \mathbb{R}, p \text{ lsc in } 0 \iff \inf_x \phi(x) = \sup_y \psi(y) \in \mathbb{R}. \tag{6.9}$$

The difference $\inf \phi - \sup \psi$ is the **duality gap**.

Proof. $\inf_x \phi(x) = p(0) \geq p^{\star\star}(0) = \sup_y \psi(y)$ so must show $p(0) \in \mathbb{R}$ and p lsc in 0 if and only if $p(0) = p^{\star\star}(0) \in \mathbb{R}$.

($\Rightarrow$) follows as $p^{\star\star}(0) \leq \operatorname{cl} p(0) \leq p(0)$, so $\liminf_{y\to 0} p(y) = \operatorname{cl} p(0) = p(0) \in \mathbb{R}$, so p is lsc in 0.

($\Leftarrow$) follows from the claim that if it holds then $\operatorname{cl} p$ is proper lsc convex. Convexity, lsc is clear, and an improper convex lsc function is always constant ∞ or $-\infty$, contradiction $p(0) \in \mathbb{R}$. So $(p^\star)^\star(0) = ((\operatorname{cl} p)^\star)^\star(0) = \operatorname{cl} p(0) = p(0)$. □

Theorem. Assume $f : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ is proper, lsc, and convex. Then we have the **primal-dual optimality conditions**,

$$(0, y') \in \partial f(x', 0) \tag{6.10}$$

$$\iff \{x' \in \arg\min_x \phi(x), \quad y' \in \arg\max_y \psi(y), \inf_x \psi(x) = \sup_y \psi(y)\} \tag{6.11}$$

$$\iff (x', 0) \in \partial f^\star(0, y'). \tag{6.12}$$

The set of **primal-dual optimal points** (x', y') satisfying this equation is either empty or equal to $(\arg\min \phi) \times (\arg\max \psi)$.

Proof. Follows from invertibility of subgradient in terms of conjugate functions, showing $f(x', 0) = -f^\star(0, y') \iff \phi(x') = \psi(y') \in \mathbb{R}$, and equality with infinite value is explicitly excluded by definition of arg min. □

Theorem. Assume $f : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ is proper, lsc, and convex. Then

(i) $0 \in \operatorname{int} \operatorname{dom} p$ or $0 \in \operatorname{int} \operatorname{dom} q$ implies $\inf_x \phi(x) = \sup_y \psi(y)$. ($S'$)

(ii) $0 \in \operatorname{int} \operatorname{dom} p$ and $0 \int \operatorname{int} \operatorname{dom} q$ implies $\inf_x \phi(x) = \sup_y \psi(y) \in \mathbb{R}$. ($S$)

(iii) $0 \in \int \operatorname{dom} p$ and $\inf_x \phi(x) \in \mathbb{R}$ if and only if $\arg\max_y \psi(y)$ is nonempty and bounded. (P)

(iv) $0 \in \int \operatorname{dom} q$ and $\sup_y \psi(y) \in \mathbb{R}$ if and only if $\arg\min_x \phi(x)$ is nonempty and bounded. (D)

In particular, if any of S, P, D hold, then strong duality holds - $\inf \phi = \sup \psi \in \mathbb{R}$. If S, or (P and D) hold, then there exists x', y' satisfying the primal-dual optimality conditions. Also, P implies $\partial p(0) = \arg\max_y \psi(y)$, and D implies $\partial q(0) = \arg\min_x \phi(x)$.

Proof. (i): If $p(0) = -\infty$, then $p^{\star\star}(0) \leq p(0) = -\infty$ Use $\operatorname{cl} p = p$ on $\int \operatorname{dom} p$, so $\sup \psi = \inf \psi = -\infty$. Otherwise, $p(0) \in \mathbb{R}$ so as $\operatorname{cl} p = p$ on $\int \operatorname{dom} p$, we have p is lsc in 0 and so $p^{\star\star}(0) = p(0) \in \mathbb{R}$.

This follows symmetrically on $f'(x, y) = f^\star(y, x)$.

If both $0 \in \int \operatorname{dom} p, 0 \in \int \operatorname{dom} q$, then $+\infty > p(0) \geq p^{\star\star}(0) = \sup \psi = -q(0) > -\infty$, which is finite.

Nonemptyness and boundedness follows from $0 \in \int \operatorname{dom} p$ if and only if ψ is proper lsc convex and level bounded.

Subdifferential: if (iii) then $p(0) \in \mathbb{R}$, so $\operatorname{cl} p(0) = p(0) \in \mathbb{R}$, $\operatorname{cl} p$ is then proper and lsc convex, so $\partial(\operatorname{cl} p)(0) = \arg\max \psi$, but $\partial p(0) = \partial(\operatorname{cl} p)(0)$. □

Theorem. Assume $k : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ are both proper, lsc, convex, and $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. For $f(x, u) = \langle c, x\rangle + k(x) + h(Ax - b + u)$, the primal and dual problems are of the form

$$\inf \phi(x), \phi(x) = \langle c, x\rangle + k(x) + h(Ax - b) \tag{6.13}$$

$$\sup_y \psi(y), \psi(y) = -\langle b, y\rangle - h^\star(y) - k^\star(-A^T y - c) \tag{6.14}$$

with

$$\int \operatorname{dom} p = \int (\operatorname{dom} h - A \operatorname{dom} k) + b \tag{6.15}$$

$$\int \operatorname{dom} q = \int (\operatorname{dom} k^\star - (-A^T) \operatorname{dom} h^\star) + c \tag{6.16}$$

and optimality conditions

$$\{-A^T y' - c \in \partial k(x'), y' \in \partial h(Ax' - b)\} \tag{6.17}$$

$$\iff \{x' \in \arg\min_x \phi(x), y' \in \arg\max_y \psi(y), \inf_x \phi(x) = \sup_y \psi(y)\} \tag{6.18}$$

$$\iff \{Ax' - b \in \partial h^\star(y'), x' \in \partial k^\star(-A^T y' - c)\} \tag{6.19}$$

Theorem. Assume $f : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ is proper, lsc, convex. Define the associated **Lagrangian** as $l(x, y) = -f(x, \cdot)^\star(y)$, so $l(x, y) = \inf_u(f(x, u) - \langle y, u\rangle)$. Then $l(\cdot, y)$ is convex for every y, $-l(x, \cdot)$ is lsc and convex for every x, y and $f(x, \cdot) = (-l(x, \cdot))^\star$, and $(v, y) \in \partial f(x, u) \iff v \in \partial_x l(x, y)$ and $u \in \partial_y(-l)(x, y)$.

Proof. Consider $g(x, y, u) = f(x, u) - \langle y, u\rangle$, which is proper lsc convex. Then $l(\cdot, y) = \inf_u g(\cdot, y, u)$ is convex.Then $f_x(y) = f(x, y)$, then $-l(x, \cdot) = f_x^\star(\cdot)$ so f_x is either $+\infty$ or proper lsc convex, and $-l(x, \cdot)$ is either $-\infty$ or proper lsc convex, but always lsc convex.

By subgradient definition, $(v, y) \in \partial f(x, u) \iff \inf_{u'} f(x', u') - \langle y, u'\rangle \geq f(x, u) - \langle y, u\rangle + \langle v, x' - x\rangle \forall x'$, which evaluated at $x' = x$ implies $\inf_{u'} f(x, u') - \langle y, u'\rangle = f(x, u) - \langle y, u\rangle$.

Continuing, we obtain $(v, y) \in \partial f(x, u) \iff y \in \partial_u f(x, u), \in \partial_x l(x, y)$, and the first condition is equivalent to $u \in \partial f_x^\star(y)$, or $u \in \partial(-l(x, \cdot))^{\star\star} = \partial_y(-l)(x, y)$ as required. □

Definition. *For any function $l : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ we say that (x', y') is a **saddle point** of l if $l(x, y') \geq l(x', y') \geq l(x', y)$ for all x, y. The set of all saddle points is denoted by $\operatorname{sp} l$.*

Evquivalently, $(x', y') \in \operatorname{sp} l$ if $\inf_x l(x, y') = l(x', y') = \sup_y l(x', y)$.

Theorem. Assume f is proper, lsc, and convex with associated Lagrangian l. Then $\phi(x) = \sup_y l(x, y)$, and $\psi(y) = \inf_x l(x, y)$, and the primal problem is $\inf_x \phi(x) = \inf_x \sup_y l(x, y)$, and the dual problem is $\sup_y \psi(y) = \sup_y \inf_x l(x, y)$. Moreover, the optimality condition

$$\{x' \in \arg\min_x \phi(x), y' \in \arg\max_y \psi(y), \inf_x \phi(x) = \sup_y \psi(y)\} \tag{6.20}$$

$$\iff (x', y') \in \operatorname{sp} l \tag{6.21}$$

$$\iff \{0 \in \partial_x l(x', y'), 0 \in \partial_y(-l)(x', y')\} \tag{6.22}$$

Theorem. Assume $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ are nonempty, closed, convex, and $L : X \times Y \to \mathbb{R}$ is a continuous function with $L(\cdot, y)$ is convex for every y and $-L(x, \cdot)$ convex for every x. Then $l(x, y) = L(x, y) + \delta_X(x) - \delta_Y(y)$ with the convention $+\infty - \infty = +\infty$ on the right, is the Lagrangian to $f(x, u) = \sup_y l(x, y) + \langle u, y\rangle = (-l(x, \cdot))^\star(u)$.

f is proper, lsc, and convex, so the previous result applies with primal and dual problems $\phi(x) = \delta_X(x) + \sup_{y\in Y} L(x, y)$, $\psi(y) = -\delta_Y(y) + \int_{x\in X} L(x, y)$. Moreover, if X and Y are bounded, then $\operatorname{sp} l$ is nonempty and bounded.

Proof. For a fixed y, $\inf_x l(x, y) = -f^\star(0, y) = \psi(y)$, and for a fixed x, $\sup_y l(x, y) = f(x, 0) = f_x^{\star\star}(0) = f(x, 0) = \phi(x)$ as f_x is either proper lsc convex or $+\infty$.

The optimality condition is equivalent to $(0, y') \in f(x', 0) \iff 0 \in \partial_x l(x', y'), 0 \in \partial_y(-l)(x', y')$, which is the saddle point condition. □

7. Numerical Optimality

Definition. *For $\phi : \mathbb{R}^n \to \overline{\mathbb{R}}$, a point x is an ϵ-optimal solution if $\phi(x) - \inf \phi \leq \epsilon$.*

Definition. *Assume (x^k, y^k) is a primal-dual feasible pair - so $x^k \in \operatorname{dom} \phi$ and $y^k \in \operatorname{dom} \psi$. Then $\phi(x^k) \geq \psi(y^k)$, and $0 \leq \phi(x^k) - \inf \phi \leq \phi(x^k) - \psi^{y^k} = \gamma(x^k, y^k) := \gamma$. γ is the **numerical primal-dual gap**. If $\gamma < \epsilon$ then x^k is an ϵ-optimal solution with **optimality certificate** y^k.*

The normalized gap is $\overline{\gamma} = \overline{\gamma}(x^k, y^k) = \frac{\phi(x^k) - \psi(y^k)}{\psi(y^k)}$.

Definition. *Assume $\phi(x) = \phi_0(x) = \sum_{i=1}^{n_p} \delta_{g_i(x)\leq 0}$, $\psi(y) = \psi_0(y) - \sum_{i=1}^{n_d} \delta_{h_i(x)\leq 0}$ where $\operatorname{dom} \phi_0 = \operatorname{dom} \psi_0 = \mathbb{R}^n$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $h_i : \mathbb{R}^m \to \mathbb{R}$ are suitable continuous real-valued convex functions, so the primal and dual constraints are of the form $g_i(x) \leq 0, h_i(y) \leq 0$. Then the primal and dual infeasibilities are defined as $\eta_p = \max\{0, g_1(x^k), \ldots, g_{n_p}(x^k)\}$ and $\eta_d = \max\{0, h_1(y^k), \ldots, h_{n_d}(y^k)\}$.*

8. First-Order Methods

Definition. *For $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we define*

*(i) The **forward step**, $F_{\tau_k f}(x^k) = (I - \tau_k \partial f)x^k$*

*(ii) The **backward step**, $B_{\tau_k f}(x^k) = (I + \tau_k \partial f)^{-1} x^k$.*

Theorem. If $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is proper lsc convex with $\tau > 0$, then the backward step is $B_{\tau f}(x) = \arg\min_y\{\frac{1}{2}\|y - x\|_2^2 + \tau f(y)\}$ and is therefore unique.

Proof. $y \in B_{\tau f}(x) \iff 0 \in y - x + \tau \partial f(y) \iff y \in \arg\min_{y'}\{\frac{1}{2}\|y' - x\|_2^2 + \tau f(y')\}$ □

Theorem. Assume f is proper lsc convex and $\arg\min f \neq \emptyset$. The **forward step** is $x^{k+1} \in F_{\tau_k f}(x^k)$. The sequence is not unique, can get stuck if x^k is infeasible.

The **backward step** is $x^{k+1} = B_{\tau_k f}(x^k)$ - which is a **unique** sequence, and cannot get stuck. Sub-steps are as hard as the original problem (but strictly convex).

(i) Forward stepping: $x^{k+1} \in F_{\tau_k f}(x^k)$:
(ii) Backward stepping: $x^{k+1} = B_{\tau_k f}(x^k)$.

If $f = g + h, \partial f = \partial g + \partial h$ with f, g, h proper lsc convex, $\arg\min f \neq \emptyset$, we can do:

(i) Backward-Backward stepping: $x^{k+1} = B_{\tau_k h} B_{\tau_k g}(x^k)$.
(ii) Forward-Backward stepping: $x^{k+1} \in B_{\tau_k h} F_{\tau_k g}(x^k)$. If $f(x) = g(x) + \delta_C(x)$, g differentiable, $C \neq \emptyset$ closed and convex, then $x^{k+1} \in \arg\min_x\{\frac{1}{2}\|y - (x^k - \tau_k \Delta g(x^k))\|_2^2 + \delta_C(x)\} = \Pi_C(x^k - \tau_k \Delta g(x^k))$, which is a gradient projection.

9. Interior-Point Methods

Definition. *For a cone K we define the* ***canonical barriers*** *$F = F_K$ and associated parameters θ_F.*

(i) $K = K_n^{LP} = \{x \in \mathbb{R}^n | x_1, \dots, x_n \geq 0\}$, $F(x) = \sum_{i=1}^n -\log x_i$, $\theta_F = n$.
(ii) $K = K_n^{SOCP} = \{x \in \mathbb{R}^n | x_n \geq \sqrt{x_1^2 + \cdots + x_{n-1}^2}\}$, $F(x) = -\log(x_n^2 - x_1^2 - \cdots - x_{n-1}^2)$, $\theta_F = 2$.
(iii) $K = K_n^{SDP} = \{X \in \mathbb{R}^{n \times n} | X$ *symmetric positive semidefinite*$\}$, $F(x) = -\log\det X$, $\theta_F = n$.
(iv) $K = K^1 \times K^2$, *then* $F_K(x^1, x^2) = F_{K^1}(x^1) + F_{K^2}(x^2)$, *with* $\theta_F = \theta_{F^1} + \theta_{F^2}$.

Theorem. If F is a canonical barrier for K, then F is smooth on $\operatorname{dom} F = \int K$ and strictly convex, $F(tx) = F(x) - \theta_F \log t$ for all $x \in \operatorname{dom} F$, and for $x \in \operatorname{dom} F$, we have

(i) $-\nabla F(x) \in \operatorname{dom} F$
(ii) $\langle \nabla F(x), x \rangle = -\theta_F$,
(iii) $-\nabla F(-\nabla F(x)) = x$,
(iv) $-\nabla F(tx) = -\frac{1}{t} \nabla F(x)$

Proof. Differentiate with respect to t and let $t = 1$. □

Theorem. Consider the problem $\inf \langle c, x \rangle$ s.t. $Ax - b \geq_K 0$. The dual problem is $\sup \langle -b, y \rangle$ s.t. $-A^T y = c$, $y \geq_{K^\star} 0$. Replacing y by $-y$ and assuming K is self-dual $K^\star = K$ we obtain the dual as $\sup \langle b, y \rangle$ such that $A^T y = c, y \geq_K 0$.

The **primal central path** is the mapping

$$t \mapsto x(t) = \arg\min\{-t\langle b, y \rangle + F(y) + \delta_{A^T y = c}\} \tag{9.1}$$

The **dual central path** is the mapping $t \mapsto y(t) = \arg\min\{-t\langle b, y \rangle + F(y) + \delta_{A^T y = c}\}$

The **primal-dual central path** is the mapping $t \mapsto z(t) = (x(t), y(t))$ for some $t > 0$, if and only if

$$Ax - b \in \operatorname{dom} F \tag{9.2}$$

$$A^T y = c \tag{9.3}$$

$$ty + \nabla F(Ax - b) = 0 \tag{9.4}$$

Proof. $y = -\frac{1}{y} \Delta f(Ax - b) \in \operatorname{dom} F$ as $Ax - b \in \operatorname{dom} F$ and $\operatorname{dom} F$ is a cone. We also have that if x is on the primal path, then $A^T y = c$, so y is feasible.

For dual optimality, we need $0 \in -tb + \nabla F(y) + N_{A^T y = c}$. As $-tb + \nabla F(y) = -tAx \in \operatorname{range} A$, y is the unique dual solution. Multiplying the dual optimality result by A^T gives the optimality condition for the primal central point. □

Theorem. For feasible x, y (so $Ax - b \in K$, $A^T y = c$, $y \in K$), the duality gap is $\phi(x) - \psi(y) = \langle y, Ax - b \rangle$. Moreover, for points $(x(t), y(t))$ on the central path, the duality gap is $\phi(x(t)) - \psi(y(t)) = \frac{\theta_F}{t}$.

Proof.

$$\phi(x) - \psi(y) = \langle c, x \rangle - \langle b, y \rangle \tag{9.5}$$

$$= \langle y, Ax - b \rangle \tag{9.6}$$

$$= \left\langle -\frac{1}{t} \nabla F(Ax(t) - b), Ax(t) - b \right\rangle \tag{9.7}$$

$$= \frac{\theta_F}{t}. \tag{9.8}$$

□

Theorem. We define $\|v\|_x^\star = (v^T \nabla^2 F(Ax - b)^{-1} v)^{\frac{1}{2}}$, $z = (x, y)$, so $z(t)$ is the primal-dual central path, and $\operatorname{dist}(z, z(t)) = \|ty + \nabla F(Ax - b)\|_x^\star$. Then for $Ax - b \in \operatorname{dom} F$, $y \in \operatorname{dom} F$, $A^T y = c$, we have $\operatorname{dist}(z, z(t)) \leq 1$ implies $\phi(x) - \psi(y) \leq 2(\phi(x(t)) - \psi(y(t))) = 2\frac{\theta_F}{t}$.

Proof. If we linearize $\nabla F(Ax^{k+1} - b) = \nabla F(Ax^k - b + A\Delta x) \approx \nabla F(Ax^k - b) + \nabla^2 F(Ax^k - b) A \Delta x$, and solve the constraints $A^T(y) = c, ty + \nabla F(Ax - b) = 0$. □

Theorem. Assume $0 < \rho \leq \kappa < \frac{1}{10}$, $t^k > 0$ fixed, and $z^k = (x^k, y^k)$ strictly feasible, so $Ax^k - b \in \operatorname{dom} F$, $y^k \in \operatorname{dom} f$, such that $\operatorname{dist}(z^k, z(t^k)) < \kappa$.

If we apply a full Newton step with $\tau_k = 1$ and $t^{k+1} = (1 + \frac{\rho}{\sqrt{\theta_F}}) t^k$ to generate z^{k+1}, then x^{k+1}, y^{k+1} are strictly primal and dual feasible, and $\operatorname{dist}(z^{k+1}, z(t^{k+1})) > \kappa$ as well.

10. Support Vector Machines

Definition. *The primal formulation of an SVM is*

$$\inf_{w,b} \frac{1}{2}\|w\|_2^2 \tag{10.1}$$

such that $1 \leq y^i(\langle w, x^i \rangle + b)$ *for all* $1 \leq i \leq n$.

The dual formulation is

$$\inf_{z \in \mathbb{R}^n} \frac{1}{2}\|\sum_{i=1}^n y^i x^i z_i\|_2^2 + e^T z \tag{10.2}$$

such that $z_i \leq 0$, $\sum_{i=1}^n y^i z_i = 0$.

11. Total Variation and Applications

Definition. *For $u \in L^1(\Omega, \mathbb{R}^m)$, the* ***total variation*** *of u is defined as*

$$TV(u) = \sup_{v \in C_c^1(\Omega, \mathbb{R}^{m \times n}), \|v\|_\infty \leq 1} \int_\Omega \langle u, \operatorname{div} v \rangle dx \tag{11.1}$$

Theorem. Assume $A \subseteq \Omega$ is a set so that its boundary is C^1 and satisfies $\mathcal{H}^{n-1}(\Omega \cap \partial A) < \infty$. Define

$$1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \tag{11.2}$$

then $TV(1_A) = \mathcal{H}^{n-1}(\Omega \cap \partial A)$.

Proof. The lower bound follows from

$$TV(1_A) = \sup_{v \in C_c^1(\Omega, \mathbb{R}^n), \|v\|_\infty \leq 1} \int_{\partial A} \langle v, n \rangle ds \tag{11.3}$$

by Gauss's theorem. □

Theorem (Coarea formula)**.** If $u \in BV(\Omega)$, then $TV(1_{u(x) > t}) < \infty$ for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, and $TV(u) = \int_\mathbb{R} TV(1_{u > t}) dt$.

Definition. *For $\Omega \subseteq \mathbb{R}^d$ and $k \geq 1$, define the space $BV^k(\Omega)$ as $BV^k = \{u \in W^{k-1,1} | \nabla^{k-1} u \in BV(\Omega, R^{d^{k-1}})\}$ and the higher-order total variation as*

$$TV^k(u) = \sup_{v \in C_c^k(\Omega, \operatorname{Sym}^k(\mathbb{R}^d)), \|v\|_\infty \leq 1} \int_\Omega u \operatorname{div}^k v dx = TV(\nabla^{k-1} u) \tag{11.4}$$

12. Relaxation

Definition. *Consider the* ***Chan-Vese*** *model*

$$f_{CV}(C, c_1, c_2) = \int_C (g - c_1^2 dx + \int_{\Omega \setminus C} = (g - c_2)^2 dx + \lambda \mathcal{H}^{d-1}(C)). \tag{12.1}$$

Theorem. Let c_1, c_2 be fixed, and consider $\inf_{u: \Omega \to [0,1], u \in BV(\Omega)} f(u), f(u) = \langle u, s \rangle_{L^1} + \lambda TV(u)$. Then if u is a minimizer of f, and $u(x) \in \{0, 1\} a.e.$, then C is a minimizer of $f_{CV(\cdot, c_1, c_2)}$.

Proof. Follows by definitions - must have $u = 1_C$. □

Definition. *Let $\mathcal{C} = BV(\Omega, [0,1]) = \{u \in BV(\Omega) | u(x) \in [0,1] a.e.\}$. Then for $u \in \mathcal{C}$, $\alpha \in [0,1]$. Define $\overline{u}_\alpha = 1_{\{u > \alpha\}}$. Then $f: \mathcal{C} \to \mathbb{R}$ satisfies the* ***generalized coarea condition*** *if and only if*

$$f(u) = \int_0^1 f(\overline{u}_\alpha) d\alpha \tag{12.2}$$

for all $u \in \mathcal{C}$.

Theorem. Let $f * u = TV(u)$, the condition is the cooarea formula. As the condition is additive, we need only show $\int_\Omega s(x) u(x) = \int_0^1 \int_\Omega s(x) 1_{\{u(x) > \alpha\}} d\alpha$, where we use Fubini due to $s \in L^\infty(\Omega)$.

Theorem. Assume $f: \mathcal{C} \to \mathbb{R}$ satisfies the generalized coarea condition, and $u^\star$ satisfies $u^\star \in \arg\min_{u \in \mathcal{C}} f(u)$. Then for almost every $\alpha \in [0,1]$, the thresholded function satisfies $\overline{u}_\alpha^\star \in \arg\min_{u \in BV(\Omega, \{0,1\})} f(u)$.

Proof. Follows by considering the set $S_\epsilon = \{\alpha \in [0,1] | f(u^\star) \leq f(u^\star_\alpha) - \epsilon\}$ for some $\epsilon > 0$, and showing that this implies $f(u^\star \leq \int_0^1 f(\overline{u}^\star_\alpha))d\alpha - \epsilon L^1(S_\epsilon)$ which contradicts the generalized coarea formula. $\square$

References