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Contents

1

Conditional Expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Ω is a set, $\mathcal{F}$ is a σ-algebra on Ω, and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$.

Definition 1.1. $\mathcal{F}$ is a σ-algebra on Ω if it satisfies

- $\emptyset, \Omega \in \mathcal{F}$
- $A \in \mathcal{F} ==> A^c \in \mathcal{F}$
- $(A_n)_{n\geq 0}$ is a collection of sets in $\mathcal{F}$ then $\cup_n A_n \in \mathcal{F}$.

Definition 1.2. $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ if

- $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a set function.
- $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$,
- $(A_n)_{n\geq 0}$ is a collection of pairwise disjoint sets in $\mathcal{F}$, then $\mathbb{P}(\cup_n A_n) = \sum_n \mathbb{P}(A_n)$.

Definition 1.3. The Borel σ-algebra $\mathbb{B}(\mathbb{R})$ is the σ-algebra generated by the open sets of $\mathbb{R}$. Call $\mathcal{O}$ the collection of open subsets of $\mathbb{R}$, then

$$\mathbb{B}(\mathbb{R}) = \cap\{\xi : \xi \text{ is a sigma algebra containing } \mathcal{O}\} \tag{1.1}$$

Definition 1.4. $\mathcal{A}$ a collection of subsets of Ω, then we write $\sigma(\mathcal{A}) = \cap\{\xi : \xi \text{ a sigma algebra containing } \mathcal{A}\}$

Definition 1.5. X is a random variable on $(\Omega, \mathcal{F})$ if $X : \Omega- > \mathbb{R}$ is a function with the property that $X^{-1}(V) \in \mathcal{F}$ for all V open sets in $\mathbb{R}$.

Exercise 1.6. *If X is a random variable then $\{B \subseteq \mathbb{R}, X^{-1}(B) \in \mathcal{F}\}$ is a σ-algebra and contains $\mathbb{B}(\mathbb{R})$.*

If $(X_i, i \in I)$ is a collection of random variables, then we write $\sigma(X_i, i \in I) = \sigma(\{\omega \in \Omega : X_{i(\omega) \in \mathcal{B}\}, i \in I, B \in \mathbb{B}(\mathbb{R})))}$ and it is the smallest σ-algebra that makes all the X_i's measurable.

Definition 1.7. First we define it for the positive simple random variables.

$$\mathbb{E}\left(\sum_{i=1}^{n} c_i \mathbf{1}(A_i)\right) = \sum_{i=1}^{n} \mathbb{P}(A_i). \tag{1.2}$$

with c_i positive constants, $(A_i) \in \mathcal{F}$.

We can extend this to any positive random variable $X \geq 0$ by approximation X as the limit of piecewise constant functions.

For a general X, we write $X = X^+ - X^-$ with $X^+ = \max(X, 0), X^- = \max(-X, 0)$.

If at least one of $\mathbb{E}(X^+)$ or $\mathbb{E}(X^-)$ is finite, then we define $\mathbb{E}(X) = \mathbb{E}(X^+) + \mathbb{E}(X^-)$.

We call X integrable if $\mathbb{E}(|X|) < \infty$.

Definition 1.8. Let $A, B \in \mathcal{F}, \mathbb{P}(B) > 0$. Then

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$
$$\mathbb{E}[X|B] = \frac{\mathbb{E}(X\mathbb{1}(B))}{\mathbb{P}(B)}$$

Goal - we want to define $\mathbb{E}(X|\mathcal{G})$ that is a random variable measurable with respect to the σ-algebra $\mathcal{G}$.

1.1 *Discrete Case*

Suppose $\mathcal{G}$ is a σ-algebra countably generated $(B_{i)_{i \in \mathbb{N}}}$ is a collection of pairwise disjoint sets in $\mathcal{F}$ with $\cup B_i = \Omega$. Let $\mathcal{G} = \sigma(B_i, i \in \mathbb{N})$.

It is easy to check that $\mathcal{G} = \{\cup_{j \in J} B_j, J \subseteq N\}$.

Let X be an integrable random variable. Then

$$X' = \mathbb{E}(X|\mathcal{G}) = \sum_{i \in \mathbb{N}} \mathbb{E}(X|\mathcal{B}_i)\, \mathbb{I}(B_i)$$

(i) X' is $\mathcal{G}$-measurable (check).

(ii)

$$\mathbb{E}\big(|X'|\big) \leq \mathbb{E}(|X|) \tag{1.3}$$

and so X' is integrable.

(iii) $\forall G \in \mathcal{G}$, then

$$\mathbb{E}\big(X\mathbb{I}(G) = \mathbb{E}\big(X'\mathbb{I}(G)\big)\big) \tag{1.4}$$

(check).

1.2 Existence and Uniqueness

Definition 1.9. $A \in \mathcal{F}$, A happens almost surely (a.s.) if $\mathbb{P}(A) = 1$.

Theorem 1.10 (Monotone Convergence Theorem)**.** *If $X_n \geq 0$ is a sequence of random variables and $X_n \uparrow X$ as $n \to \infty$ a.s, then*

$$\mathbb{E}(X_n) \uparrow \mathbb{E}(X) \tag{1.5}$$

almost surely as $n \to \infty$.

Theorem 1.11 (Dominated Convergence Theorem)**.** *If (X_n) is a sequence of random variables such that $|X_n| \leq Y$ for Y an integrable random variable, then if $X_n \overset{as}{\to} X$ then $\mathbb{E}(X_n) \overset{as}{\to} \mathbb{E}(X)$.*

Definition 1.12. For $p \in [1, \infty)$, f measurable functions, then

$$\|f\|_p = E[|f|^p]^{\frac{1}{p}} \tag{1.6}$$

$$\|f\|_\infty = \inf\{\lambda : |f| \leq \lambda a.e.\} \tag{1.7}$$

Definition 1.13.

$$L^p = L^p(\Omega, \mathcal{F}, \mathbb{P}) = \{f : \|f\|_p < \infty\}$$

Formally, L^p is the collection of equivalence classes where two functions are equivalent if they are equal a.e. We will just represent an element of L^p by a function, but remember that equality is a.e.

Theorem 1.14. *The space $(L^2, \|\cdot\|_2)$ is a Hilbert space with $\langle U, V\rangle >= \mathbb{E}(UV)$.*

Suppose $\mathcal{H}$ is a closed subspace, then $\forall f \in L^2$ there exists a unique $g \in \mathcal{H}$ such that $\|f-g\|_2 = \inf\{\|f-h\|_2, h \in \mathcal{H}$ and $\langle f-g, h\rangle = 0$ for all $h \in \mathcal{H}$.

We call g the orthogonal projection of f onto $\mathcal{H}$.

Theorem 1.15. *Let $(\Omega, \mathcal{F}, \P)$ be an underlying probability space, and let X be an integrable random variable, and let $\mathcal{G} \subset \mathcal{F}$ sub σ-algebra. Then there exists a random variable Y such that*

(i) Y is $\mathcal{G}$-measurable

(ii) If $A \in \mathcal{G}$,

$$\mathbb{E}(X\mathbb{I}(A) = \mathbb{E}(Y\mathbb{I}(A))) \tag{1.8}$$

and Y is integrable.

Moreover, if Y' also satisfies the above properties, then $Y = Y'$ a.s.

Remark 1.16. *Y is called a version of the conditional expectation of X given $\mathcal{G}$ and we write $\mathcal{G} = \sigma(Z)$ as $Y = \mathbb{E}(X|\mathcal{G})$.*

Remark 1.17. *(b) could be replaced by the following condition: for all Z $\mathcal{G}$-measurable, bounded random variables,*

$$\mathbb{E}(XZ) = \mathbb{E}(YZ) \tag{1.9}$$

Proof. **Uniqueness** - let Y' satisfy (a) and (b). If we consider $\{Y' - Y > 0\} = A$, A is $\mathcal{G}$ measurable. From (b),

$$\mathbb{E}\big((Y'-Y)\mathbb{I}(A) = \mathbb{E}(X\mathbb{I}(A))\big) - \mathbb{E}(X\mathbb{I}(A)) = 0$$

and hence $\mathbb{P}(Y' - Y > 0)) = 0$ which implies that $Y' \leq Y$ a.s. Similarly, $Y' \geq Y$ a.s.

Existence - Complete the following three steps:

(i) $X \in L^2(\Omega, \mathcal{F}, \P)$ is a Hilbert space with $\langle U, V\rangle = \mathbb{E}(UV)$. The space $L^2(\Omega, \mathcal{G}, \P)$ is a closed subspace.

$$X_n \to X(L^2) => X_n \xrightarrow{p} X => \exists subseq X_{n_k} \xrightarrow{as} X => X' = \limsup X_{n_k} \tag{1.10}$$

We can write

$$\begin{aligned} L^2(\Omega, \mathcal{F}, \P) &= L^2(\Omega, \mathcal{G}, \P) + L^2(\Omega, \mathcal{G}, \P)^{\perp} \\ X &= Y + Z \end{aligned}$$

Set $Y = \mathbb{E}(X|\mathcal{G})$, Y is $\mathcal{G}$-measurable, $A \in \mathcal{G}$.

$$\mathbb{E}(X\mathbb{I}(A)) = EY\mathbb{I}(A) + \underbrace{EZ\mathbb{I}(A)}_{=0}$$

(ii) If $X \geq 0$ then $Y \geq 0$ a.s. Consider $A = \{Y < 0\}$, then

$$0 \leq \mathbb{E}(X\mathbb{I}(A)) = \mathbb{E}(Y\mathbb{I}(A)) \leq 0 \tag{1.11}$$

Thus $\mathbb{P}(A) = 0 \Rightarrow Y \geq 0$ a.s.

Let $X \geq 0$, Set $0 \leq X_n = max(X, n) \leq n$, so $X_n \in L^2$ for all n. Write $Y_n = \mathbb{E}(X_n|\mathcal{G})$, then $Y_n \geq 0$ a.s., Y_n is increasing a.s.. Set $Y = \limsup Y_n$. So Y is $\mathcal{G}$-measurable. We will show $Y = \mathbb{E}(X|\mathcal{G})$ a.s. For all $A \in \mathcal{G}$, we need to check $\mathbb{E}(X\mathbb{I}(A)) = \mathbb{E}(Y\mathbb{I}(A))$. We know that $\mathbb{E}(X_n\mathbb{I}(A)) = \mathbb{E}(Y_n\mathbb{I}(A))$, and $Y_n \uparrow Y$ a.s. Thus, by monotone convergence theorem, $\mathbb{E}(X\mathbb{I}(A)) = \mathbb{E}(Y\mathbb{I}(A))$.

If X is integrable, setting $A = \Omega$, we have Y is integrable.

(iii) X is a general random variable, not necessarily in L^2 or ≥ 0. Then we have that $X = X^+ + X^-$. We define $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X^+|\mathcal{G}) - \mathbb{E}(X^-|\mathcal{G})$. This satisfies (a), (b).

□

Remark 1.18. *If $X \geq 0$, we can always define $Y = \mathbb{E}(X|\mathcal{G})$ a.s. The integrability condition of Y may not be satisfied.*

Definition 1.19. Let $\mathcal{G}_0, \mathcal{G}_1, \ldots$ be sub σ-algebras of $\mathcal{F}$. Then they are called independent if for all $i, j \in \mathbb{N}$,

$$\mathbb{P}\left(G_i \cap \cdots \cap G_j\right) = \Pi_{i=1}^{n} \mathbb{P}(G_i) \tag{1.12}$$

Theorem 1.20. *(i) If $X \geq 0$ then $\mathbb{E}(X|\mathcal{G}) \geq 0$*

(ii) $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$ *($A = \Omega$)*

(iii) *X is $\mathcal{G}$-measurable implies $\mathbb{E}(X|\mathcal{G}) = X$ a.s.*

(iv) *X is independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$.*

Theorem 1.21 (Fatau's lemma)**.** *$X_n \geq 0$, then for all n,*

$$\mathbb{E}(\liminf X_n) \leq \liminf \mathbb{E}(X_n) \tag{1.13}$$

Theorem 1.22 (Conditional Monotone Convergence)**.** *Let $X_n \geq 0$, $X_n \uparrow X$ a.s. Then*

$$\mathbb{E}(X_n|\mathcal{G}) \uparrow \mathbb{E}(X|\mathcal{G}) \text{ a.s.} \tag{1.14}$$

Proof. Set $Y_n = \mathbb{E}(X_n|\mathcal{G})$. Then $Y_n \geq 0$ and Y_n is increasing. Set $Y = \limsup Y_n$. Then Y is $\mathcal{G}$-measurable. □

Theorem 1.23 (Conditional Fatau's Lemma)**.** *$X_n \geq 0$, then*

$$\mathbb{E}(\liminf X_n|\mathcal{G}) \leq \liminf \mathbb{E}(X_n|\mathcal{G}) \text{ a.s.} \tag{1.15}$$

Proof. Let X denote the limit inferior of the X_n. For every natural number k define pointwise the random variable $Y_k = \inf_{n \geq k} X_n$. Then the sequence $Y_1, Y_2, \dots$ is increasing and converges pointwise to X. For $k \leq n$, we have $Y_k \leq X_n$, so that

$$\mathbb{E}(Y_k|\mathcal{G}) \leq \mathbb{E}(X_n|\mathcal{G}) \text{ a.s} \tag{1.16}$$

by the monotonicity of conditional expectation, hence

$$\mathbb{E}(Y_k|\mathcal{G}) \leq \inf_{n \geq k} \mathbb{E}(X_n|\mathcal{G}) \text{ a.s.} \tag{1.17}$$

because the countable union of the exceptional sets of probability zero is again a null set. Using the definition of X, its representation as pointwise limit of the Y_k, the monotone convergence theorem for conditional expectations, the last inequality, and the definition of the limit inferior, it follows that almost surely

$$\begin{aligned}
\mathbb{E}\left(\liminf_{n\to\infty} X_n|\mathcal{G}\right) &= \mathbb{E}(X|\mathcal{G}) && (1.18)\\
&= \mathbb{E}\left(\lim_{k\to\infty} Y_k|\mathcal{G}\right) && (1.19)\\
&= \lim_{k\to\infty} \mathbb{E}(Y_k|\mathcal{G}) && (1.20)\\
&\leq \lim_{k\to\infty} \inf_{n\geq k} \mathbb{E}(X_n|\mathcal{G}) && (1.21)\\
&= \liminf_{n\to\infty} \mathbb{E}(X_n|\mathcal{G}) && (1.22)
\end{aligned}$$

□

Theorem 1.24 (Conditional dominated convergence)**.** *TODO*

1.3 Conditional Jensen's Inequalities

Let X be an integrable random variable such that $\phi(x)$ is integrable of ϕ is non-negative. Suppose $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra. Then

$$\mathbb{E}(\phi(X)|\mathcal{G}) \geq \phi(\mathbb{E}(X|\mathcal{G})) \tag{1.23}$$

almost surely. In particular, if $1 \leq p < \infty$, then

$$\|\mathbb{E}(X|\mathcal{G})\|_p \leq \|X\|_p \tag{1.24}$$

Proof. Every convex function can be written as $\phi(x) = \sup_{i\in\mathbb{N}}(a_i x + b_i), a_i, b_i \in \mathbb{R}$. Then

$$\begin{aligned}
\mathbb{E}(\phi(X)|\mathcal{G}) &\geq a\mathbb{E}(X|\mathcal{G}) + b_i\\
\mathbb{E}(\phi(X)|\mathcal{G}) &\geq \sup_{i\in\mathbb{N}}(a_i\mathbb{E}(X|\mathcal{G}) + b_i)\\
&= \phi(\mathbb{E}(X|\mathcal{G})
\end{aligned}$$

The second part follows from

$$\|\mathbb{E}(X|\mathcal{G})\|_p^p = \mathbb{E}(|\mathbb{E}(X|\mathcal{G})|^p) \leq \mathbb{E}(\mathbb{E}(|X|^p|\mathcal{G})) = \mathbb{E}(|X|^p) = \|X\|_p^p \tag{1.25}$$

□

Proposition 1.25 (Tower Property)**.** *Let $X \in L^1$, $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ be* *sub-σ-algebras. Then*

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})\,|\mathcal{H}) = \mathbb{E}(X|\mathcal{H}) \tag{1.26}$$

almost surely.

Proof. Clearly $\mathbb{E}(X|\mathcal{H})$ is $\mathcal{H}$-measurable. Let $A \in \mathcal{H}$. Then

$$\mathbb{E}(\mathbb{E}(X|\mathcal{H})\,\mathbb{I}(A)) = \mathbb{E}(X\mathbb{I}(A)) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})\,\mathbb{I}(A)) \tag{1.27}$$

□

Proposition 1.26. *Let $X \in L^1$, $\mathcal{G} \subset \mathcal{F}$ be sub-σ-algebras. Suppose that Y is bounded, $\mathcal{G}$-measurable. Then*

$$\mathbb{E}(XY|\mathcal{G}) = Y\mathbb{E}(X|\mathcal{G}) \tag{1.28}$$

almost surely.

Proof. Clearly $Y\mathbb{E}(X|\mathcal{G})$ is $\mathcal{G}$-measurable. Let $A \in \mathcal{G}$. Then

$$\mathbb{E}(Y\mathbb{E}(X|\mathcal{G})\,\mathbb{I}(A)) = \mathbb{E}\left(\mathbb{E}(X|\mathcal{G}) \qquad \underbrace{(Y\mathbb{I}(A))}_{\mathcal{G}\text{-measurable, bounded}}\right) = \mathbb{E}(XY\mathbb{I}(A)) \tag{1.29}$$

□

Definition 1.27. A collection $\mathcal{A}$ of subsets of Ω is called a π-system if for all $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.

Proposition 1.28 (Uniqueness of extension)**.** *Suppose that ξ is a σ-algebra on E. Let μ_1, μ_2 be two measures on (E, ξ) that agree on a π-system generating ξ and $\mu_1(E) = \mu_2(E) < \infty$. Then $\mu_1 = \mu_2$ everywhere on ξ.*

Theorem 1.29. *Let $X \in L^1$, $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ two sub-σ-algebras. If $\sigma(X, \mathcal{G})$ is independent of $\mathcal{H}$, then*

$$\mathbb{E}(X|\sigma(\mathcal{G}, \mathcal{H})) = \mathbb{E}(X|\mathcal{G}) \tag{1.30}$$

almost surely.

Proof. Take $A \in \mathcal{G}, B \in \mathcal{H}$.

$$\begin{aligned}
\mathbb{E}(\mathbb{E}(X|\mathcal{G})\,\mathbb{I}(A)\,\mathbb{I}(B)) = \mathbb{P}(B)\,\mathbb{E}(\mathbb{E}(X|\mathcal{G})\,\mathbb{I}(A)) \\
&= \mathbb{P}(B)\,\mathbb{E}(X\mathbb{I}(A)) \\
&= \mathbb{E}(X\mathbb{I}(A)\,\mathbb{I}(B)) \\
&= \mathbb{E}(\mathbb{E}(X|\sigma(\mathcal{G},\mathcal{H}))\,\mathbb{I}(A\cap B))
\end{aligned}$$

Assume $X \geq 0$, the general case follows by writing $X = X^+ - X^-$.

Now, letting $F \in \mathcal{F}$, we have that $\mu(F) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})\,\mathbb{I}(F))$, and if μ, ν are two measures on $(\Omega, p\mathcal{F})$, setting $\mathcal{A} = \{A \cap B, A \in \mathcal{G}, B \in \mathcal{H}\}$. Then $\mathcal{A}$ is a π-system.

μ, ν are two measurables that agree on the π-system $\mathcal{A}$ and $\mu(\Omega) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X) = \nu\Omega < \infty$, since X is integrable. Note that $\mathcal{A}$ generates $\sigma(\mathcal{G},\mathcal{H})$.

So, by the uniqueness of extension theorem, μ, ν agree everywhere on $\sigma(\mathcal{G},\mathcal{H})$. □

Remark 1.30. *If we only had X independent of $\mathcal{H}$ and $\mathcal{G}$ independent of $\mathcal{H}$, the conclusion can fail. For example, consider coin tosses X, Y independent 0, 1 with probability $\frac{1}{2}$, and $Z = \mathbb{I}(X = Y)$.*

1.4 *Product Measures and Fubini's Theorem*

Definition 1.31. A measure space (E, ξ, μ) is called σ-finite if there exists sets $(S_n)_n$ with $\cup S_n = E$ and $\mu(S_n) < \infty$ for all n.

Let (E_1, ξ_1, μ_1) and (E_2, ξ_2, μ_2) be two σ-finite measure spaces, with $\mathcal{A} = \{A_1 \times A_2 : A_1 \in \xi_1, A_2 \in \xi_2\}$ a π-system of subsets of $E = E_1 \times E_2$. Define $\xi = \xi_1 \otimes \xi_2 = \sigma(A)$.

Definition 1.32 (Product measure)**.** Let (E_1, ξ_1, μ_1) and (E_2, ξ_2, μ_2) be two σ-finite measure spaces. Then there exists a unique measure μ on (E, ξ) ($\mu = \mu_1 \otimes \mu_2$) satisfying

$$\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2) \tag{1.31}$$

for all $A_1 \in \xi_1, A_2 \in \xi_2$.

Theorem 1.33 (Fubini's Theorem)**.** *Let (E_1, ξ_1, μ_1) and (E_2, ξ_2, μ_2) be σ-finite measure spaces. Let $f \geq 0$, f is ξ-measurable. Then*

$$\mu(f) = \int_{E_1} \left(\int_{E_2} f(x_1, x_2) \mu_2(dx_2) \right) \mu_1(dx_1) \tag{1.32}$$

If f is integrable, then $x_2 \mapsto f(x_1, x_2)$ is u_2-integrable for u_1-almost all x.

Moreover, $x_1 \mapsto \int_{E_2} f(x_1, x_2 \mu_2(dx_2)$ is μ_1-integrable and $\mu(f)$ is given by (1.32).

1.5 Examples of Conditional Expectation

Definition 1.34. A random vector $(X_1, X_2, \ldots, X_n) \in \mathbb{R}^n$ is called a Gaussian random vector if and only if for all $a_1, \ldots, a_n \in \mathbb{R}$,

$$a_1 X_1 + \cdots + a_n X_n \tag{1.33}$$

is a Gaussian random variable.

$(X_t)_{t \geq 0}$ is called a Gaussian process if for all $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, the vector $X_{t_1}, \ldots, X_{t_n}$ is a Gaussian random vector.

Example 1.35 (Gaussian case)**.** *Let (X, Y) e a Gaussian vector in $\mathbb{R}^2$. We want to calculate*

$$\mathbb{E}(X|Y) = \mathbb{E}(X|\sigma(Y)) = X' \tag{1.34}$$

where $X' = f(Y)$ with f a Borel function. Let's try f of a linear function $X' = aY = b, a, b \in \mathbb{R}$ to be determined.

Note that $\mathbb{E}(X) = \mathbb{E}(X')$ and $E(X' - X)Y = 0 \Rightarrow Cov(X - X', Y) = 0$ by laws of conditional expectation. Then we have that

$$a\mathbb{E}(Y) + b = \mathbb{E}(X)\, Cov(X, Y) = a\mathbb{V}(X) \tag{1.35}$$

TODO - continue inference

1.6 Notation for Example Sheet 1

(i) $G \vee H = \sigma(G, H)$.

(ii) Let X, Y be two random variables taking values in R with joint density $f_{X,Y}(x, y)$ and $h : \mathbb{R} \to \mathbb{R}$ be a Borel function such that

$h(X)$ is integrable. We want to calculate

$$\mathbb{E}(h(X)|Y) = \mathbb{E}(h(X)|\sigma(Y)) \tag{1.36}$$

Let g be bounded and measurable. Then

$$\begin{aligned}
\mathbb{E}(h(X)g(Y)) &= \int\int h(x)g(y)f_{X,Y}(x,y)dxdy && (1.37)\\
&= \int\int h(x)g(y)\frac{f_{X,Y}(x,y)}{f_Y(y)}f_Y(y)dxdy && (1.38)\\
&= \int\left(\int h(x)\frac{f_{X,Y}(x,y)}{f_Y(y)}dx\right)g(y)f_Y(y)dy && (1.39)
\end{aligned}$$

with $0/0 = 0$

Set $\phi(y) = \int h(x)\frac{f_{X,Y}(x,y)}{f_Y(y)}dx$ if $f_Y(y) > 0$, and 0 otherwise. Then we have

$$\mathbb{E}(h(X)|Y) = \phi(Y) \tag{1.40}$$

almost surely, and

$$\mathbb{E}(h(X)|Y) = \int h(x)\nu(Y, dx) \tag{1.41}$$

with $\nu(y, dx) = \frac{f_{X,Y}(x,y)}{f_Y(y)}\mathbb{I}(f_Y(y) > 0)\,dx = f_{X|Y}(x|y)dx$.

$\nu(y, dx)$ is called the conditional distribution of X given $Y = y$ and $f_{X|Y}(x|y)$ is the conditional density of X given $Y = y$.

2

Discrete Time Martingales

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and (E, ξ) a measurable space. Usually $E = \mathbb{R}, \mathbb{R}^d, \mathbb{C}$. For us, $E = \mathbb{R}$. A sequence $X = (X_n)_{n\geq 0}$ of random variables taking values in E is called a **stochastic process**.

A **filtration** is an increasing family $(\mathcal{F}_n)_{n\geq 0}$ of sub-σ-algebras of $\mathcal{F}_n$, so $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$.

Intuitively, $\mathcal{F}_n$ is the information available to us at time n. To every stochastic process X we associate a filtration called the natural filtration

$$(\mathcal{F}_n^X)_{n\geq 0}, \mathcal{F}_n^X = \sigma(X_k, k \leq n) \tag{2.1}$$

A stochastic process X is called adapted to $(\mathcal{F}_n)_{n\geq 0}$ if X_n is $\mathcal{F}_n$-measurable for all n.

A stochastic process X is called integrable if X_n is integrable for all n.

Definition 2.1. An adapted integrable process $(X_n)_{n\geq 0}$ taking values in $\mathbb{R}$ is called a

(i) **martingale** if $\mathbb{E}(X_n|\mathcal{F}_m) = X_m$ for all $n \geq m$.

(ii) **super-martingale** if $\mathbb{E}(X_n|\mathcal{F}_m) \leq X_m$ for all $n \geq m$.

(iii) **sub-martingale** if $\mathbb{E}(X_n|\mathcal{F}_m) \geq X_m$ for all $n \geq m$.

Remark 2.2. *A (sub,super)-martingale with respect to a filtration $\mathcal{F}_n$ is also a (sub, super)-martingale with respect to the natural filtration of X_n (by the tower property)*

Example 2.3. *Suppose* (ξ_i) *are* IID *random variables with* $\mathbb{E}(\xi_i) = 0$*. Set* $X_n = \sum_{i=1}^n \xi_i$*. Then* (X_n) *is a martingale.*

Example 2.4. *As above, but let* (ξ_i) *be* IID *with* $\mathbb{E}(\xi_i) = 1$*. Then* $X_n = \Pi_{i=1}^n \xi_i$ *is a martingale.*

Definition 2.5. A random variables $T : \Omega \to \mathbb{Z}_+ \cup \{\infty\}$ is called a stopping time if $\{T \leq n\} \in \mathcal{F}_n$ for all n. Equivalently, $\{T = n\} \in \mathcal{F}_n$ for all n.

Example 2.6. *(i) Constant times are trivial stopping times.*

(ii) $A \in \mathcal{B}(\mathbb{R})$*. Define* $T_A = \inf\{n \geq 0 | X_n \in A\}$*, with* $\inf \emptyset = \infty$*. Then* T_A *is a stopping time.*

Proposition 2.7. *Let* $S, T, (T_n)$ *be stopping times on the filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P})$*. Then* $S \wedge T, S \vee T, \inf_n T_n, \liminf_n T_n, \limsup_n T_n$ *are stopping times.*

Notation. *T stopping time, then* $X_T(\omega) = X_{T(\omega)}(\omega)$*. The stopped process* X^T *is defined by* $X_t^T = X_{T \wedge t}$*.*

$\mathcal{F}_T = \{A \in \mathcal{F} | A \cap T \leq T \in \mathcal{F}_t, \forall t\}$.

Proposition 2.8. $(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P})$*,* $X = (X_n)_{n \geq 0}$ *is adapted.*

(i) $S \leq T$*, stopping times, then* $\mathcal{F}_S \subseteq \mathcal{F}_T$

(ii) $X_T \mathbb{I}(T < \infty)$ *is* $\mathcal{F}_T$*-measurable.*

(iii) T a stopping time, then X^T *is adapted*

(iv) If X is integrable, then X^T *is integrable.*

Proof. Let $A \in \xi$. Need to show that $\{X_T \mathbb{I}(T < \infty) \in A\} \in \mathcal{F}_T$.

$$\{X_T \mathbb{I}(T < \infty)\} \cap \{T \leq t\} = \cup_{s \leq t} \left(\underbrace{\{T = s\}}_{\mathcal{F}_s \subseteq \mathcal{F}_t} \cap \underbrace{\{X_s \in A\}}_{\in \mathcal{F}_s \subseteq \mathcal{F}_t} \right) \in \mathcal{F}_t \tag{2.2}$$

□

2.1 Optional Stopping

Theorem 2.9. *Let X be a martingale.*

(i) *If T is a stopping time, then X^T is also a martingale. In particular, $\mathbb{E}(X_{T\wedge t}) = \mathbb{E}(X_0)$ for all t.*

(ii)

(iii)

(iv)

Proof. By the tower property, it is sufficient to check

$$\begin{aligned}\mathbb{E}(X_{T\wedge t}|\mathcal{F}_{t-1}) &= \mathbb{E}\left(\sum_{i=1}^{t-1} X_s \underbrace{\mathbb{I}(T=s)}_{\in\mathcal{F}_s\subseteq\mathcal{F}_{t-1}} |\mathcal{F}_{t-1}\right) + \mathbb{E}(X_t\mathbb{I}(T>t-1)\,|\mathcal{F}_{t-1}) \\ &= \sum_{s=0}^{t-1}\mathbb{I}(T=s)\,X_s + \mathbb{I}(t>t-1)\,X_{t-1} = X_{T\wedge(t-1)}\end{aligned}$$

Since it is a martingale, $\mathbb{E}(X_{T\wedge t}) = \mathbb{E}(X_0)$. □

Theorem 2.10. *Let X be a martingale.*

(i) *If T is a stopping time, then X^T is also a martingale, so in particular*

$$\mathbb{E}(X_{T\wedge t}) = \mathbb{E}(X_0) \tag{2.3}$$

(ii) *If $X \le T$ are bounded stopping times, then $\mathbb{E}(X_T|\mathcal{F}_S) = X_S$ almost surely.*

Proof. Let $S \le T \le n$. Then $X_T = (X_T - X_{T-1}) + (X_{T-1} - X_{T-2}) + \cdots + (X_{S+1} - X_S) + X_S = X_s + \sum_{k=0}^{n}(X_{k+1} - X_k)\mathbb{I}(S \le k < T)$.

Let $A \in \mathcal{F}_s$. Then

$$\mathbb{E}(X_T\mathbb{I}(A)) = \mathbb{E}(X_s\mathbb{I}(A)) + \sum_{k=0}^{n}\mathbb{E}((X_{k+1} - X_k)\mathbb{I}(S\le k<T)\,\mathbb{I}(A)) \tag{2.4}$$

$$= \mathbb{E}(X_s\mathbb{I}(A)) \tag{2.5}$$

□

Remark 2.11. *The optimal stopping theorem also holds for super/sub-martingales with the respective martingale inequalities in the statement.*

Example 2.12. *Suppose that* $(\xi_i)_i$ *are random variables with*

$$\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = \frac{1}{2} \tag{2.6}$$

Set $X_0 = 0, X_n = \sum_{i=1}^n \xi_i$. *This is a simply symmetric random walk on* X_n.

Let $T = \inf\{n \geq 0 : X_n = 1\}$. *Then* $\P T < \infty = 1$, *but* T *is not bounded.*

Proposition 2.13. *If* X *is a positive supermartingale and* T *is a stopping time which is finite almost surely* ($\mathbb{P}(T < \infty) = 1$)*, then*

$$\mathbb{E}(X_T) \leq \mathbb{E}(X_0) \tag{2.7}$$

Proof.

$$\mathbb{E}(X_T) = \mathbb{E}\left(\liminf_{t \to \infty} X_{t \wedge T}\right) \leq \liminf_{t \to \infty} \mathbb{E}(X_{t \wedge T}) \leq \mathbb{E}(X_0) \tag{2.8}$$

□

2.2 *Hitting Probabilities for a Simple Symmetric Random Walk*

Let (ξ_i) be IID ± 1 equally likely. Let $X_0 = 0, X_n = \sum_{i=1}^n \xi_i$. For all $x \in Z$ let

$$T_x = \inf\{n \geq 0 : X_n = x\} \tag{2.9}$$

which is a stopping time. We want to explore hitting probabilities $(\mathbb{P}(T_{-a} < T_b))$ for $a, b > 0$. If $\mathbb{E}(T) < \infty$, then by (iv) in Theorem 2.10, $\mathbb{E}(X_T) = \mathbb{E}(X_0) = 0$.

$$\mathbb{E}(X_T) = -a\mathbb{P}(T_{-a} < T_b) + b\mathbb{P}(T_b < T_{-a}) = 0 \tag{2.10}$$

and thus obtain that

$$\mathbb{P}(T_{-a} < T_b) = \frac{b}{a+b}. \tag{2.11}$$

Remains to check $\mathbb{E}(T) < \infty$. We have $\mathbb{P}(\xi_1 = 1, \xi_{a+b} = 1) = \frac{1}{2^{a+b}}$.

2.3 Martingale Convergence Theorem

Theorem 2.14. *Let $X = (X_n)_{n \geq 0}$ be a (super-)-martingale bounded in L^1, that is, $\sup_{n \geq 0} \mathbb{E}(|X_n|) < \infty$. Then X_n converges as $n \to \infty$ almost surely towards an a.s. finite limit $X \in L^1(\mathcal{F}_\infty)$ with $\mathcal{F}_\infty = \sigma(\mathcal{F}_n, n \geq 0)$. To prove it we will use Doob's trick which counts up-crossings of intervals with rational endpoints.*

Corollary 2.15. *Let X be a positive supermartingale. Then it converges to an almost surely finite limit as $n \to \infty$.*

Proof.

$$\mathbb{E}(|X_n|) = \mathbb{E}(X_n) \leq \mathbb{E}(X_0) < \infty \tag{2.12}$$

□

Proof. Let $x = (x_n)_n$ be a sequence of real numbers, and let $a < b$ be two real numbers. Let $T_0(x) = 0$ and inductively for $k \geq 0$,

$$S_{k+1}(x) = \inf\{n \geq T_k(x) : x_n \leq a\} T_{k+1}(x) = \inf\{n \geq S_{k+1}(x) : x_n \geq b\} \tag{2.13}$$

with the usual convention that $\inf \emptyset = \infty$.

Define $N_n([a,b], x) = \sup\{k \geq 0 : T_k(x) \leq n\}$ - the number of up-crossings of the interval $[a,b]$ by the sequence x by the time n. As $n \to \infty$, we have

$$N_n([a,b], x) \uparrow N([a,b], x) = \sup\{k \geq 0 : T_k(x) < \infty\}, \tag{2.14}$$

the total number of up-crossings of the interval $[a,b]$. □

Lemma 2.16. *A sequence of rationals $x = (x_n)_n$ converges in $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ if and only if $N([a,b], x) < \infty$ for all rationals a, b.*

Proof. Assume x converges. Then if for some $a < b$ we had that $N([a,b], x) = \infty$, then $\liminf_n x_n \leq a < b \leq \limsup_n x_n$, which is a contradiction.

Then, suppose that x does converge. Then $\liminf_n x_n > \limsup_n x_n$, and so taking a, b rationals between these two numbers gives that $N([a,b], x) = \infty$ as required. □

Theorem 2.17 (Doob's up-crossing inequality)**.** *Let X be a supermartingale and $a < b$ be two real numbers. Then for all $n \geq 0$,*

$$(b-a)\mathbb{E}(N_n([a,b],X)) \leq \mathbb{E}\big((X_n - a)^-\big) \tag{2.15}$$

Proof. For all k,

$$X_{T_k} - X_{S_k} \geq b - a \tag{2.16}$$

□

2.4 *Uniform Integrability*

Theorem 2.18. *Suppose $X \in L^1$. Then the collection of random variables*

$$\{\mathbb{E}(X|\mathcal{G})\} \tag{2.17}$$

for $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra is uniformly integrable.

Proof. Since $X \in L^1$, for all $\epsilon > 0$ there exists $S > 0$ such that if $A \in \mathcal{F}$ and $\mathbb{P}(A) < \delta$, then $\mathbb{E}(|X|\mathbb{I}(A)) \leq \epsilon$.

Set $Y = \mathbb{E}(X|\mathcal{G})$. Then $\mathbb{E}(|Y|) \leq \mathbb{E}(|X|)$. Choose $\lambda < \infty$ such that $\mathbb{E}(|X|) \leq \lambda\delta$. Then

$$\mathbb{P}(|Y| \geq \lambda) \leq \frac{\mathbb{E}(|Y|)}{\lambda} \leq \delta \tag{2.18}$$

by Markov's inequality.

Then

$$\mathbb{E}(|Y|\mathbb{I}(|Y| \geq \lambda)) \leq \mathbb{E}(\mathbb{E}(|X||\mathcal{G})\,\mathbb{I}(|Y| \geq \lambda)) \tag{2.19}$$

$$= \mathbb{E}(|X|\mathbb{I}(|Y| \geq \lambda)) \tag{2.20}$$

$$\leq \epsilon \tag{2.21}$$

□

Definition 2.19. A process $X = (X_n)_{n \geq 0}$ is called a uniformly integrable martingale if it is a martingale and the collection (X_n) is uniformly integrable.

Theorem 2.20. *Let X be a martingale. Then the following are equivalent.*

(i) *X is a uniformly integrable martingale.*

(ii) *X converges almost surely and in L^1 to a limit X_∞ as $n \to \infty$.*

(iii) *There exists a random variable $Z \in L^1$ such that $X_n = \mathbb{E}(Z|\mathcal{F}_n)$ almost surely for all $n \geq 0$.*

Theorem 2.21 (Chapter 13 of Williams)**.** *Let $X_n, X \in L^1$ for all $n \geq 0$ and suppose that $X_n \xrightarrow{as} X$ as $n \to \infty$. Then X_n converges to X in L^1 if and only if (X_n) is uniformly integrable.*

Proof. We proceed as follows.

$(i) \Rightarrow (ii)$ Since X is uniformly integrable, it is bounded in L^1 and by the martingale convergence theorem, we get that X_n converges almost surely to a finite limit X_∞. By the previous theorem, Theorem 2.21 gives L^1 convergence.

$(ii) \Rightarrow (iii)$ Set $Z = X_\infty$. We need to show that $X_n = \mathbb{E}(Z|\mathcal{F}_n)$ almost surely for all $n \geq 0$. For all $m \geq n$ by the martingale property we have

$$\|X_n - \mathbb{E}(X_\infty|\mathcal{F}_n)\|_1 = \|\mathbb{E}(X_m - X_\infty|\mathcal{F}_n)\|_1 \leq \|X_m - X_\infty\|_1 \to 0 \tag{2.22}$$

as $m \to \infty$.

$(iii) \Rightarrow (i)$ $\mathbb{E}(Z|\mathcal{F}_n)$ is a martingale by the tower property of conditional expectation. Uniform integrability follows from Theorem 2.18.

□

Remark 2.22. *If X is UI then $X_\infty = \mathbb{E}(Z|\mathcal{F}_\infty)$ a.s where $F_\infty = \sigma(\mathcal{F}_n, n \geq 0)$.*

Remark 2.23. *If X is a super/sub-martingale UI, then it converges almost surely and in L^1 to a finite limit X_∞ with $\mathbb{E}(X_\infty|\mathcal{F}_n)\,(\geq)(\leq)X_n$ almost surely.*

Example 2.24. *Let $X_1, X_2, \ldots$ be IID random variables with $\mathbb{P}(X = 0) = \mathbb{P}(X = 2) = \frac{1}{2}$. Set $Y_n = X_1 \cdot \cdots \cdot X_n$. Then Y_n is a martingale.*

As $\mathbb{E}(Y_n) = 1$ for all n, we have (Y_n) is bounded in L^1, and it converges almost surely to 0. But $\mathbb{E}(Y_n) = 1$ for all n, and hence it does not converge in L^1.

If X is a UI martingale and T is a stopping time, then we can unambiguously define

$$X_T = \sum_{n=0}^{\infty} X_n \mathbb{I}(T = n) + X_\infty \mathbb{I}(T = \infty) \tag{2.23}$$

Theorem 2.25 (Optional stopping for UI martingales)**.** *Let X be a UI martingale and let S, T be stopping times with $S \leq T$. Then*

$$\mathbb{E}(X_T | \mathcal{F}_S) = X_S \tag{2.24}$$

almost surely.

Proof. We first show that $\mathbb{E}(X_\infty | \mathcal{F}_T) = X_T$ almost surely for any stopping time T. First, check that $X_T \in L^1$. Since $|X_n| \leq \mathbb{E}(|X_\infty| | \mathcal{F}_n)$, we have

$$\mathbb{E}(|X_T|) = \sum_{n=0}^{\infty} \mathbb{E}(|X_n| \mathbb{I}(T = n) + \mathbb{E}(|X_\infty| \mathbb{I}(T = \infty))) \tag{2.25}$$

$$\leq \sum_{n \in \mathbb{Z}^+ \cup \{\infty\}} \mathbb{E}(|X_\infty| \mathbb{I}(T = n)) \tag{2.26}$$

$$= \mathbb{E}(|X_\infty|) \tag{2.27}$$

Let $B \in \mathcal{F}_T$. Then

$$\mathbb{E}(\mathbb{I}(B)\, X_T) = \sum_{n \in \mathbb{Z}^+ \cup \{\infty\}} \mathbb{E}(\mathbb{I}(B)\, \mathbb{I}(T = n)\, X_n) \tag{2.28}$$

$$= \sum_{n \in Z^+ \cup \{\infty\}} \mathbb{E}(\mathbb{I}(B)\, \mathbb{I}(T = n)\, X_\infty) \tag{2.29}$$

$$= \mathbb{E}(\mathbb{I}(B)\, X_\infty) \tag{2.30}$$

where for the second equality we used that $\mathbb{E}(X_\infty | \mathcal{F}_n) = X_n$ almost surely.

Clearly $X_T is \mathcal{F}_T$-measurable, and hence $\mathbb{E}(X_\infty | \mathcal{F}_T) = X_T$ almost surely. Using the tower property of conditional expectation, we have for stopping times $S \leq T$ (as $\mathcal{F}_S \subseteq \mathcal{F}_T$),

$$\mathbb{E}(X_T|\mathcal{F}_S) = \mathbb{E}(\mathbb{E}(X_\infty|\mathcal{F}_T)\,|\mathcal{F}_S) \tag{2.31}$$
$$= \mathbb{E}(X_\infty|\mathcal{F}_S) \tag{2.32}$$
$$= X_S \tag{2.33}$$

almost surely. □

2.5 *Backwards Martingales*

Let ... $\subseteq \mathcal{G}_{-2} \subseteq G_{-1} \subseteq G_0$ be a sequence of ....

Fill in proof from lecture notes

2.6 *Applications of Martingales*

Theorem 2.26 (Kolmogrov's $0-1$ law). *Let $(X_i)_{i\geq 1}$ be a sequence of IID random variables. Let $\mathcal{F}_n = \sigma(X_k, k \geq n)$ and $\mathcal{F}_\infty = \cap_{n\geq 0}\mathcal{F}_n$. Then $\mathcal{F}_\infty$ is trivial - that is, every $A \in \mathcal{F}_\infty$ has probability $\mathbb{P}(A) \in \{0,1\}$.*

Proof. Let $G_n = \sigma(X_k, k \leq n)$ and $A \in \mathcal{F}_\infty$. Since $\mathcal{G}_n$ is independent of $\mathcal{F}_{n+1}$, we have that

$$\mathbb{E}(\mathbb{I}(A)\,|\mathcal{G}_n) = \mathbb{P}(A) \tag{2.34}$$

Theorem 2.26 (LN) gives that $\mathbb{P}(A) = \mathbb{E}(\mathbb{I}(A)\,|\mathcal{G}_n)$ converges to $\mathbb{E}(\mathbb{I}(A)\,|\mathcal{G}_\infty)$ as $n \to \infty$, where $\mathcal{G}_\infty = \sigma(\mathcal{G}_n, n \geq 0)$. Then we deduce that $\mathbb{E}(\mathbb{I}(A)\,|\mathcal{G}_n) = \mathbb{I}(A) = \mathbb{P}(A)$ as $\mathcal{F}_\infty \subseteq \mathcal{G}_\infty$. Therefore, $\mathbb{P}(A) =$ □

link to correct theorem

Theorem 2.27 (Strong law of large numbers). *Let $(X_i)_{i\geq 1}$ be a sequence of IID random variables in L^1 with $\mu = \mathbb{E}(X_i)$. Let $S_n = \sum_{i=1}^n X_i$ and $S_0 = 0$. Then $\frac{S_n}{n} \to \mu$ as $n \to \infty$ almost surely and in L^1.*

Proof. □

fill in, this is somewhat involved.

Theorem 2.28 (Kakutani's product martingale theorem). *Let $(X_n)_{n\geq 0}$ be a sequence of independent non-negative random variables of mean 1. Let $M_0 = 1$, $M_n = \prod_{i=1}^n X_i$ for $n \in \mathbb{N}$. Then $(M_n)_{n\geq 0}$ is a non-negative martingale and $M_n \to M_\infty$ a.s. as $n \to \infty$ for some random variable M_∞. We set $a_{n=\mathbb{E}(\sqrt{X_n})}$, then $a_n \in (0,1]$. Moreover,*

(i) If $\prod_n a_n > 0$, then $M_n \to M_\infty$ in L^1 and $\mathbb{E}(M_\infty) = 1$,

(ii) If $\prod_n a_n = 0$, then $M_\infty = 0$ almost surely.

Proof. □ fill in

2.6.1 *Martingale proof of the Radon-Nikodym theorem*

Let $\mathbb{P}, \mathbb{Q}$ be two probability measures on the measurable space $\Omega, \mathcal{F}$. Assume that $\mathcal{F}$ is countably generated, that is, there exists a collection of sets $(F_n)_{n \in \mathbb{N}}$ such that $\mathcal{F} = \sigma(F_N, n \in \mathbb{N})$. Then the following are equivalent.

(i) $\mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A)$ for all $A \in \mathcal{F}$. That is, $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ and write $\mathbb{Q} << \mathbb{P}$

(ii) For all $\epsilon > 0$, there exists $\delta > 0$ such that $\mathbb{P}(A) \leq \delta \Rightarrow \mathbb{Q}(A) \leq \epsilon$.

(iii) There exists a non-negative random variable X such that

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}(X \mathbb{I}(A)) \tag{2.35}$$

Proof. $(i) \to (ii)$. If (ii) does not hold, then there exists $\epsilon > 0$ such that for all $n \geq 1$ there exists a set A_n with $\mathbb{P}(A_n) \leq \frac{1}{n^2}$ and $\mathbb{Q}(A_n) \geq \epsilon$. By Borel-Cantelli, we get that $\mathbb{P}(A_n i.o) = 0$. Therefore from (i) we get that $\mathbb{Q}(A_n i.o) = 0$. But

$$\mathbb{Q}(A_n i.o) = \mathbb{Q}(\cap_n \cup_{k \geq n} A_k) = \lim_{n \to \infty} Q(\cup_{k \geq n} A_k) \geq \epsilon \tag{2.36}$$

which is a contradiction.

$(ii) \to (iii)$. Consider the filtration $\mathcal{F}_n = \sigma(F_k, k \leq n)$. Let

$$\mathcal{A}_n = \{H_1 \cap \cdots \cap H_n | H_i = F_i \text{ or } F_i^c\} \tag{2.37}$$

then it is easy to see that $\mathcal{F}_n = \sigma(A_n)$. Note also that sets in A_n are disjoint.

□ continue proof

3

Stochastic Processes in Continuous Time

Our setting is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space with $t \in J \subseteq \mathbb{R}_+ = [0, \infty)$

Definition 3.1. A filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing collection of σ-algebras $(\mathcal{F}_t)_{t \in J}$, satisfying $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $t \geq s$. A stochastic process in continuous time is an ordered collection of random variables on Ω.

4

Bibliography