ADVANCED FINANCIAL MODELS SUMMARY

ANDREW TULLOCH

1. Arbitrage Theory

Definition. *An investment/consumption strategy is a predictable process H satisfying the **self-financing** condition*

$$H_{t-1} \cdot P_{t-1} \geq H_t \cdot P_{t-1} \tag{1.1}$$

The corresponding consumption process c_t is given as

$$c_t = H_{t-1} \cdot P_{t-1} - H_t \cdot P_{t-1} \tag{1.2}$$

Definition. *X_t is **predictable** if X_t is $\mathcal{F}_{t-1}$-measurable for all $t \geq 1$.*

Definition. *A state price density is a strictly positive adapted process Y such that the process $Y_t P_t$ is a martingale.*

Definition. *An **absolute arbitrage** is a strategy H such that there exists a non-random time $T > 0$ with the properties*

(i) $X_0(H) = 0 = X_T(H)$ *almost surely, and*
(ii) $\mathbb{P}\left(\sum_{t=1}^{T} c_t > 0\right) > 0$.

Definition. *An asset is a* numeraire *if its price is strictly positive for all time, almost surely.*

Theorem. If a numeraire exists, then we have that if an investment/consumption strategy is an arbitrage for the market model, there exists a pure investment strategy H' and a non-random time horizon T' such that

(i) $X_0(H') = 0$,
(ii) $X_{T'}(H') \geq 0$ almost surely,
(iii) $\mathbb{P}(X_{T'}(H') > 0) > 0$.

Theorem. A market model has no arbitrage if and only if there exists a state price density.

Proof. ($\Rightarrow$) $H_0 = \mathbb{E}(YP_1)$, so if $0 \leq \mathbb{E}(YH \cdot P_1) = H \cdot \mathbb{E}(YP_1) = H \cdot P_0 = 0$, so by pigeonhole $H \cdot P_1 = 0$. ($\Leftarrow$) By separating hyperplane argument, we have $P = \{\mathbb{E}(YP_1) \,|Y > 0, \mathbb{E}(Y\|P_1\|) < \infty\}$, so either $P_0 \in P$ (and so state price density exists), or there exists H with for all $p \in P$, $H \cdot (p - P_0) \geq 0$ (with $p^\star \in P$) with $H \cdot (p^\star - P_0) > 0$.

Then setting $Y = \epsilon Y_0$, we have a pigeonhole argument showing that $P(X - H \cdot P_0) = 0$, with $X \geq 0$ a.s. ◻

Definition. *A supermartingale is an adapted integrable process such that*

$$\mathbb{E}(X_t|\mathcal{F}_s) \leq X_s \tag{1.3}$$

for all $0 \leq s \leq t$.

Definition. *A stopping time for a filtration $\mathcal{F}_t$ is a random variable τ such that the event $\{\tau \leq t\}$ is $\mathcal{F}_t$ measurable for all t.*

Definition. *For an adapted process X_t and a stopping time τ, the stopped process X^τ is given by $X_{t\wedge\tau}$.*

Theorem. Let X be a martingale and let τ be a stopping time, then X^τ is a martingale.

Proof. The process $K_t = \mathbb{I}(t \leq \tau)$ is predictable, bounded, so X^τ is a martingale transform and hence a martingale. ◻

Definition. *A local martingale is an adapted process X_t such that there exists an increasing sequence of stopping times τ_n with $\tau_n \uparrow \infty$ such that the stopped process X^{τ_n} is a martingale for each N.*

Theorem. Martingales are local martingales..

Theorem. Let X be a local martingale, with $|X_s| < Y_t$ a.s for all $0 \leq s \leq t$. If $\mathbb{E}(Y_t) < \infty$ for all $t \geq 0$, then X is a true martingale.

Proof. Conditional dominated convergence theorem, $X_{t\wedge\tau_n}$ is a martingale. ◻

Theorem. Let X be a local martingale, with $X_t \geq 0$ for all $t \geq 0$. Then X is a supermartingale.

Proof. Fatou's Lemma. ◻

Theorem. If X is a discrete-time local martingale with $X_t \geq 0$ for all $t \geq 0$. Then X is a martingale.

Theorem. The probability measure $\mathbb{Q}$ is equivalent to the measure $\mathbb{P}$ if and only if there exists a positive random variable ξ such that $\mathbb{Q}(A) = \mathbb{E}^P(\xi\mathbb{I}(A))$.

The random variable ξ is call the density, or Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$.

Definition. *A numeraire is an asset with a strictly positive price at all times.*

Definition. *An equivalent martingale measure is any probability measure $\mathbb{Q}$ equivalent opt $\mathbb{P}$ such that the discounted price process $\frac{S_t}{N_t}$ is a martingale under $\mathbb{Q}$, where N_t is the numeraire price process.*

Definition. *Let Y be a state price density, and fix a time horizon $T > 0$. Then*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{Y_T N_T}{Y_0 N_0} \tag{1.4}$$

is an equivalent martingale measure relative to N for the model.

Definition. *Suppose $\mathbb{Q}$ is an equivalent martingale measure for the market P_t. Then*

$$Y_t = \frac{N_0}{N_T}\mathbb{E}^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t\right) \tag{1.5}$$

is a state price density.

2. Pricing and Hedging Contingent Claims

Definition. *A European claim with payout ξ_T is **replicable** (or **attainable**) if there exists a pure investment strategy H such that $X_T(H) = \xi_T$ almost surely.*

Theorem. Suppose that our market has no arbitrage. Let ξ_T be the payout of a European option, and let H be the replicating strategy. Suppose that the option is priced at ξ_t for $0 \leq t \leq T$. Then if the augmented market with the option has no arbitrage,

$$\xi_t = X_t(H) \tag{2.1}$$

for all $0 \leq t \leq T$.

Proof. First fundamental theorem applied to (P, ξ). ◻

Theorem. Suppose that the market model has no arbitrage, and let Y be a state price density process. Let ξ_T be the payout of an attainable European contingent claim with maturity date $T > 0$. Suppose the claim has price ϵ_t for $0 \leq t \leq T$ and that the augmented market with the option has no arbitrage. If either $Y_T\xi_T$ is integrable of $\xi_T \geq 0$ almost surely, then

$$\xi_t = \frac{1}{Y_t}\mathbb{E}(\xi_T Y_T|\mathcal{F}_t) \tag{2.2}$$

or (if a numeraire exists),

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{N_T Y_T}{N_0 Y_0} \tag{2.3}$$

Definition. *A market is **complete** if every European contingent claim is attainable, and **incomplete** otherwise.*

Theorem. An arbitrage-free market model is complete if and only if there exists a unique state price density Y such that $Y_0 = 1$.

Proof. Uniqueness follows from $Y = Y'$, then considering $\xi = \mathbb{I}(Y_T > Y'_T)$ and pigeonhole. ◻

Theorem. Suppose the adapted process ξ_t specifies the payout of an American claim maturing at $T > 0$. Then there exists a trading strategy H such that

(i) $X_t(H) \geq \xi_t$ for all $0 \leq qt \leq T$,
(ii) $X_{\tau^\star} = \xi_{\tau^\star}$ for some stopping time $\tau^\star$, and
(iii) $X_0(H) = \sup_{\tau \leq T} \mathbb{E}(Y_\tau \xi_\tau)$.

Theorem. Let U be a discrete-time supermartingale. Then there is a unique decomposition

$$U_t = U_0 + M_t - A_t \tag{2.4}$$

where $M$is a martingale and $A$is a predictable non-decreasing process with $M_0 = A_0 = 0$.

Proof. $M_0 = 0 = A_0$,

$$M_{t+1} = M_t + U_{t+1} - \mathbb{E}(U_{t+1}|\mathcal{F}_t) \tag{2.5}$$
$$A_{t+1} = A_t + U_t - \mathbb{E}(U_{t+1}|\mathcal{F}_t)\,. \tag{2.6}$$

and telescope. □

Definition. *Let Z_t be an integrable adapted discrete-time process. Let U_t be given by the recursion*

$$U_T = Z_T \tag{2.7}$$
$$U_t = \max(Z_t, \mathbb{E}(U_{t+1}|\mathcal{F}_t)) \tag{2.8}$$

U_t is called the Snell envelope of Z_t. It is the smallest supermartingale that dominates the process Z_t.

Theorem. Let Z_t be an integrable adapted process, with U_t its Snell envelope with Doob decomposition $U_t = U_0 + M_t - A_t$. Let $\tau^\star = \min\{t \in \{0,\ldots,T\} : A_{t+1} > 0\}$ with the convention $\tau^\star = T$ on $\{A_t = 0 \forall t\}$. Then $\tau^\star$ is a stopping time, with

$$U_{\tau^\star} = U_0 + M_{\tau^\star} = Z_{\tau^\star}. \tag{2.9}$$

Theorem. Let Z be an adapted integrable process and let U be its Snell envelope. Then

$$U_0 = \sup_{\tau \le T} \mathbb{E}(Z_\tau)\,. \tag{2.10}$$

3. Brownian Motion and Stochastic Calculus

Definition. *A Brownian motion is a collection of random variables such that*

(i) $W_0(\omega) = 0$ for all $\omega \in \Omega$,
(ii) For all $0 \le t_0 < t_1 \cdots < t_n$, $W_{t_{i+1}} - W_{t_i}$ are independent with distribution $N(0, |t_{i+1} - t_i|)$,
(iii) The sample path $t \mapsto W_t(\omega)$ is continuous for all $\omega \in \Omega$.

Definition. *A simple predictable process is an adapted process α of the form $\alpha_t(\omega) = \sum_{n=1}^N \mathbb{I}((t_{n-1}, t_n])(t) a_n(\omega)$ where a_n are bounded and $\mathcal{F}_{t_{n-1}}$-measurable for $0 \le t_0 < \cdots < t_n < \infty$.*

Define the stochastic integral by the formula

$$\int_0^\infty \alpha_s dW_s = \sum_{n=1}^N a_n (W_{t_n - W_{t_{n-1}}}) \tag{3.1}$$

Theorem (Ito's Isometry)**.** For a simple predictable integrand α, we have

$$\mathbb{E}\Big((\int_0^\infty \alpha_s dW_s)^2 \Big) = \mathbb{E}\Big(\int_0^\infty \alpha_s^2 ds \Big) \tag{3.2}$$

Definition. *If α is predictable with $\mathbb{E}(\int_0^\infty \alpha_s^2 ds) < \infty$, then $\int_0^\infty \alpha_s dW_s = \lim_k \int_0^\infty \alpha_s^{(k)dW_s}$ where the limit is in $L^2(\Omega)$ where $\alpha^{(k)}$ is a sequence of simple predictable processes converging to α in $L^2(R_+ \times \Omega)$.*

Theorem. For every predictable α such that $\mathbb{E}\Big(\int_0^t \alpha_s^2 ds\Big) < \infty$ for all $t \ge 0$, there exists a continuous martingale X such that $X_t = \int_0^\infty \alpha_s \mathbb{I}(s \le t)\, dW_s$.

Theorem. If α is an adapted continuous process then $X_t = \int_0^t \alpha_s dW_s$ is a continuous local martingale. If we have $\mathbb{E}\Big(\int_0^t \alpha_s^2 ds\Big) < \infty$ for all $t \ge 0$, then X is a true martingale.

Definition. *An Ito process X is an adapted process of the form*

$$X_t = X_0 + \int_0^t \alpha_s dW_s + \int_0^t \beta_s ds \tag{3.3}$$

where X_0 is a fixed real number and α_t, β_t are predictable real-valued processes such that $\int_0^t \alpha_s^2 ds < \infty$ and $\int_0^t |\beta_s| ds < \infty$ almost surely for all $t \ge 0$.

Theorem. Let X be an Ito process and $f : \mathbb{R} \to \mathbb{R}$ twice continuously differentiable. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\alpha_s dW_s + \int_0^t [f'(X_s)\beta_s + \frac{1}{2} f''(X_s)\alpha_s^2] ds \tag{3.4}$$

Theorem. Let X be an Ito process. There exists a continuous non-decreasing process $\langle X \rangle$ called the quadratic variation of X, such that

$$\langle X \rangle_t = \lim_N \sum_{n=1}^N (X_{\frac{nt}{N}} - X_{\frac{(n-1)t}{N}})^2 \tag{3.5}$$

for each $t \ge 0$, where the limit is in probability. If

$$dX_t = \alpha_t dW_t + \beta_t dt, \tag{3.6}$$

then

$$d\langle X \rangle_t = \alpha_t^2 dt \tag{3.7}$$

Theorem. Let $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ where $(t,x) \mapsto f(t,x)$ is continuously differentiable in t and twice-continuously differentiable in x. Then

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t) dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t, X_t) dX_t^{(i)} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t) d\Big\langle X^{(i)}, \tag{3.8}$$

Theorem. Let W_t be an m-dimensional Brownian motion, with

$$Z_t = \exp(-\frac{1}{2}\int_0^t \|\alpha_s\|^2 ds + \int_-^t \alpha_s \cdot dW_s) \tag{3.9}$$

and $Z_t$be a martingale. Let $\mathbb{Q}$ be the equivalent measure with density $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$. Then the m-dimensional process $(\hat{W}_t)$ defined by $\hat{W}_t = W_t - \int_0^t \alpha_s ds$ is a Brownian motion on $(\Omega, \mathcal{F}_T, \mathbb{Q})$.

Theorem (Novikov's Condition)**.** If $\mathbb{E}\Big(\exp(\frac{1}{2}\int_0^T \|\alpha_s\|^2 ds)\Big) < \infty$, then

$$\mathbb{E}\Big(\exp(-\frac{1}{2}\int_0^T \|\alpha_s\|^2 ds + \int_0^T \alpha_s \cdot dW_s) \Big) = 1 \tag{3.10}$$

Theorem. Let $(\Omega, \mathcal{F}, \Pi)$ be a probability space with an m-dimensional Brownian motion W and filtration $\mathcal{F}_t$ generated by W. Let X be a continuous local martingale. Then there exists a unique predictable m-dimensional process α_t such that $\int_0^t \|\alpha_s\|^2 ds < \infty$ almost surely for all $t \ge 0$ and $X_t = X_0 + \int_0^t \alpha_s \cdot dW_s$. Furthermore, if $X_t > 0$ for all $t \ge 0$ then there exists a predictable β such that $\int_0^t \|\beta_s\|^2 < \infty$ and

$$X_t = X_0 \exp(-\frac{1}{2}\int_0^T \|\beta_s\|^2 ds + \int_0^T \beta_s \cdot dW_s). \tag{3.11}$$

4. Arbitrage Theory for Continuous-Time Models

Definition. *An $(n+1)$-dimensional predictable process (H, c) such that H is P-integrable is a self-financing investment/consumption strategy if and only if $d(H_t \cdot P_t) = H_t \cdot dP_t - c_t dt$. The wealth associated with a self-financing strategy H is $X_t = H_t \cdot P_t = X_0(H) + \int_0^t H_s \cdot dP_s - \int_0^t c_s ds$.*

Definition. *A trading strategy H is L-admissible if and only if the associated wealth process $X(H)$ is such that $X_t(H) \ge -L_t$ for all $t \ge 0$ a.s. where L is a given continuous non-negative adapted process.*

Definition. *An admissible investment/consumption strategy (H, c) is called an absolute arbitrage if and only if there is a non-random time T such that $X_0(H) = 0 = X_T(H)$ a.s. and $\mathbb{P}\Big(\int_0^T c_s ds > 0\Big) > 0$.*

Definition. *A state price density is a positive Ito process Y such that YP is an n-dimensional local martingale.*

Theorem. If there exists a state price density Y such that YL is locally of class D, then there are no L-admissible absolute arbitrages.

Proof. Using the self-financing condition $dX_t = d(H_t \cdot dP_t) = H_t \cdot dP_t - c_t dt$, we obtain $d(X_t Y_t) = H_t \cdot d(Y_t P_t) - Y_t c_t dt$.

Then if YL is of class D, we can show $\mathbb{E}\Big(\int_0^T H_s \cdot d(Y_s P_s) + L_T Y_T\Big) = \mathbb{E}(L_T Y_T)$ be Fatou's, stopped local martingales, and uniform integrability. Then we show $\mathbb{E}\Big(\int_0^T Y_s c_s ds\Big) = \mathbb{E}\Big(\int_0^T H_s \cdot (dY_s P_s)\Big) \le 0$ and so $c_t = 0$ a.s. □

Definition. *A family of random variables $\mathcal{Z}$ is called uniformly integrable if and only if $\lim_{k\to\infty} \sup_{Z \in \mathcal{Z}} \mathbb{E}(|Z| \mathbb{I}(|Z| > k)) = 0$.*

Theorem. Let $Z_1, \ldots, Z_n$ be a family of integrable random variables. The following statements are equivalent:

(i) $Z_n \to Z_\infty$ in L^1, and
(ii) $(Z_n)$is uniformly integrable and $Z_n \to Z_\infty$ in probability.

Definition. *A continuous adapted process Z is of class D if the family of random variables $\{Z_\tau\}$ with τ a finite stopping time is uniformly integrable. A process is locally of class D if $\{Z_{\tau\wedge t}\}$ for τ a stopping time is uniformly integrable for each $t \ge 0$.*

If $\mathbb{E}(\sup_{0\le s\le t} |Z_s|) < \infty$ for each $t \ge 0$, then Z is locally of class D. If Z is a martingale, then Z is locally of class D.

Theorem. Suppose (H,c) is a self-financing investment/consumption strategy and let $X_t = H_t \cdot P_t = X_0 + \int_0^t H_s \cdot dP_s - \int_0^t c_s ds$. Then $d(X_tY_t) = H_t \cdot d(Y_tP_t) - Y_tc_tdt$ for any Ito process Y.

Definition. *A relative arbitrage is a pure investment strategy with wealth process X such that there is a non-random time $T > 0$ satisfying $\frac{X_T}{N_T} \geq \frac{X_0}{N_0}$ a.s and $\mathbb{P}\Big(\frac{X_T}{N_T} > \frac{X_0}{N_0}\Big) > 0$.*

Definition. *An equivalent (local) martingale measure relative to the numeraire with price N is a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that $\frac{S}{N}$ is a local martingale.*

Theorem. Let $\mathbb{Q}$ be an equivalent local martingale measure. Suppose $\frac{L}{N}$ is locally in $\mathbb{Q}$-class D. Then there are no L-admissible relative arbitrages.

Theorem. There exist continuous time markets that have relative arbitrage but no absolute arbitrage.

Theorem. Let λ be a predictable m-dimensional process such that $\int_0^t \|\lambda_s\|^2 ds < \infty$ a.s. for all $t \geq 0$ and that $\sigma_t\lambda_t = \mu_t - r_t\mathbf{1}$. Let $Y_t = Y_0 \exp(-\int_0^t (r_s + \frac{\|\lambda_s\|^2}{2})ds - \int_0^t \lambda_s \cdot dW_s)$ for a constant $Y_0 > 0$ - or equivalently, $dY_t = Y_t(-r_tdt - \lambda_t \cdot dW_t)$. Then Y is a state-price density. Furthermore, if the filtration is generated by the m-dimensional Brownian motion W, all state price densities have this form.

Proof. Show YB and YS are local martingales.

Martingale representation theorem shows that all are of the form $M_t = M_0 \exp(-\frac{1}{2}\int_0^t \|\lambda_s\|^2 ds - \int_0^t \lambda_s \cdot dW_s)$. ☐

Theorem. Suppose λ is a predictable process with $\sigma_t\lambda_t = \mu_t - r_t\mathbf{1}$. If $M_t = e^{-\frac{1}{2}\int_-^t \|\lambda_s\|^2 ds - \int_0^t \lambda_s \cdot dW_s}$ is a true martingale, then the measure $\mathbb{Q}$ defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = M_T$ is an equivalent martingale measure. In particular, the stock price dynamics are given by

$$dS_t^i = S_t^i(r_tdt + \sum_j \sigma_t^{ij} d\hat{W}_t^j) \tag{4.1}$$

where $\hat{W}_t = W_t + \int_0^t \lambda_s ds$ is a $\mathbb{Q}$ Brownian motion.

5. Hedging Contingent Claims in Continuous Time Models

Theorem. Suppose $m = d$ and the $d \times d$ matrix σ_t is invertible for all t, ω so that in particular, there is a unique (up to scaling) state price density Y of the form $dY_t = Y_t(-r_tdt - \lambda_t \cdot dW_t)$ where $\lambda_t = \sigma_t^{-1}(\mu_t - r_t\mathbf{1})$.

Let ξ_T be non-negative, $\mathcal{F}_T$-measurable, and such that ξ_TY_T is integrable. Then there exists a 0-admissible strategy H with initial cost $X_0(H) = \mathbb{E}_{Y_0}(Y_T\xi_T)$ which replicates the European claim with payout ξ_T.

Furthermore, if LY is locally of class $D$and $\tilde{H}$ is an L-admissible strategy replicating the claim, then $X_0(\tilde{H}) \geq X_0(H)$.

Definition. *The Black-Scholes model is given by the pair of equations*

$$dB_t = B_trdt \tag{5.1}$$

$$dS_t = S_t(\mu dt + \sigma dW_t) \tag{5.2}$$

Consider pricing a European option with payoff $\xi_T = g(S_T)$. The unique state price density with $Y_0 = 1$ is given by $Y_t = \exp((r - \frac{\lambda^2}{2})t - \lambda W_t$ with $\lambda = \frac{\mu - r}{\sigma}$.

Thus, the is a trading strategy H which replicates the payout with

$$X_t(H) = \frac{1}{Y_t}\mathbb{E}(Y_Tg(S_T)|\mathcal{F}_t)\,. \tag{5.3}$$

The EMM $\mathbb{Q}$ is given by the density $\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp(-\frac{\lambda^2 T}{2} - \lambda W_T)$.

Theorem. Suppose that the function $V : [0,T] \times \mathbb{R}^d \to [0, \infty)$ satisfies the PDE

$$\frac{\partial V}{\partial t} + \sum_{i=1}^d rS^i \frac{\partial V}{\partial S^i} + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d a_{i,j}S^iS^j \frac{\partial^2 V}{\partial S^i \partial S^j} = rV \tag{5.4}$$

and $V(T,S) = g(S)$.

Then there exists a 0-admissible strategy H such that $X_t(H) = V(t, S_t)$. In particular, this strategy replicates the contingent claim with payout $g(S_T)$.

Furthermore, if $H = (\phi, \pi)$, then the strategy can be calculated as

$$\pi_t = \nabla V(t, S_t) = (\frac{\partial V}{\partial S_1}(t, S_t), \ldots, \frac{\partial V}{\partial S^d}(t, S_t)) \tag{5.5}$$

and $\phi_t = \frac{V(t,S_t) - \pi_t \cdot S_t}{B_t}$.

Proof. Ito's formula on $dV(t, S_t)$, and show $V(t, S_t) = \phi_tB_t + \pi_tS_t$, and $dV(t, S_t) = \phi_tdB_t = \pi_t \cdot dS_t$, so $H = (\phi, \pi)$ is a self-financing strategy that replicates $V(t, S_t)$ as required. ☐

Theorem. Suppose that $C_0(T,K) = \mathbb{E}_{e^{-rT}(S_T-K)^+}(\mathbb{Q})$. Then $\frac{\partial C_0}{\partial T}(T,K) + rK\frac{\partial C_0}{\partial K}(T,K) = \frac{\sigma(T,K)^2}{2}K^2\frac{\partial^2 C_0}{\partial K^2}(T,K)$.

Theorem. Assume that a banker hedges an option assuming constant volatility, and delta hedges with wealth evolving with $dX_t = r(X_t - \pi_tS_t) + \pi_tS_t$, with $\pi_t = V_S(t, S_t, \hat{\sigma})$. If the true dynamics are $dS_t = S_t(\mu dt + \sigma_tdW_t)$, then using the fact that V solves the BS PDE and that $dV_t = rVdt + \pi_t(dS_t - rS_tdt) + \frac{1}{2}S_t^2(\sigma_t^2 - \hat{\sigma}^2)V_{SS}dt$, we obtain

$$X_T - g(S_T) = \frac{1}{2}\int_0^T e^{r(T-t)}(\hat{\sigma}^2 - \sigma_t^2)S_t^2V_{SS}(t, S_t, \hat{\sigma})dt \tag{5.6}$$

6. Interest Rate Models

Definition. *A zero-coupon bond with maturity T is a European contingent claim that pays one unit of currency at time T. $P(t,T)$ is the price at time $t \in [0,T]$ of the bond.*

The yield $y(t,T)$ is defined by $y(t,T) = -\frac{1}{T-t}\log P(t,T)$.

The forward rate $f(t,T)$ is defined by $f(t,T) = -\frac{\partial}{\partial T}\log P(t,T)$.

Note that $P(t,T) = e^{-(T-t)y(t,T)} = e^{-\int_t^T f(t,s)ds}$.

Theorem. Let $dB_t = B_tr_tdt$ where r_t is the sport interest rate. Then there is no arbitrage relative to the numeraire if there exists an equivalent measure $\mathbb{Q}$ such that the discounted bond price process $\frac{P(t,T)}{B_t}_{t \in 0,T}$ is a local martingale for all $T > 0$. IN particular, there is no arbitrage if $P(t,T) = \mathbb{E}_{\exp(-\int_t^T r_sds)|\mathcal{F}_t}(\mathbb{Q})$ for all $0 \leq t \leq T$.

Definition. *The Vasicek model is $dr_t = \lambda(\bar{r} - r_t)dt + \sigma d\hat{W}_t$.*

We have $\mathbb{E}(\mathbb{Q})\,(r_t) = e^{-\lambda t}r_0 + (1 - e^{-\lambda t})\bar{r}$, $\mathbb{V}^{\mathbb{Q}}(r_t) = \int_0^t e^{-2\lambda(t-s)}\sigma^2 ds = \frac{\sigma^2}{2\lambda}(1 - e^{-2\lambda t})$.

Indeed, we can deduce that

$$f(t, t+x) = r_te^{-\lambda x} + \bar{r}(1 - e^{-\lambda x}) - \frac{\sigma^2}{2\lambda^2}(1 - e^{-\lambda x})^2 \tag{6.1}$$

Theorem. Consider now where the short rate is Markovian, and so $dr_t = \alpha(t, r_t)dt + \beta(t, r_t)d\hat{W}_t$ for non-random function α, β.

If we fix $T > 0$ and let $V : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfy the PDE

$$\frac{\partial V}{\partial t}(t,r) + \alpha(t,r)\frac{\partial V}{\partial r}(t,r) + \frac{1}{2}\beta(t,r)^2\frac{\partial^2 V}{\partial r^2}(t,r) = rV(t,r) \tag{6.2}$$

with $V(T,r) = 1$. Assume $P(t,T) = V(t, r_t)$. Then the discounted price process $\exp(-\int_-^t r_sds)P(t,T)$ is a $\mathbb{Q}$-local martingale.

References