ANDREW TULLOCH

ADVANCED FINANCIAL MODELS

TRINITY COLLEGE
THE UNIVERSITY OF CAMBRIDGE

Contents

1

Discrete Time Models

1.1 Standing Assumptions

(i) Zero dividends

(ii) Zero tick size

(iii) Zero transaction costs

(iv) Infinitely divisible transactions

(v) No short-selling constraints

(vi) No bid-ask spread

(vii) No market impact (infinitely deep market)

1.2 Setup

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 1.1. A random variable is a measurable map $X : \Omega \to \mathbb{R}$

Definition 1.2. A stochastic process $Y = (Y_t)_{t \in I}$ is a collection of random variables. For us, $I = \{0, 1, \dots\}$ or $[0, \infty)$.

Definition 1.3. A filtration $\mathbb{F} = (\mathcal{F})_{t \geq 0}$ is a collection of sub-σ-algebras on $\mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $0 \leq s \leq t$ (discrete and continuous time).

Example 1.4. *Tossing coins.*

(i) $\Omega = \{HH, HT, TH, TT\}$

(ii) $\mathcal{F}$ *is all 16 subsets of* Ω

(iii) $\mathbb{P}(A) = \frac{|A|}{4}$

Possible filtration

(i) $\mathcal{F}_0 = \{\emptyset, \Omega\}$

(ii) $\mathcal{F}_1 = \{\emptyset, \Omega, \{HH, HT\}, \{TH, TT\}\}$

(iii) $\mathcal{F}_2 = \mathcal{F}$

Definition 1.5. A process Y is adapted if and only if Y_t is $\mathcal{F}_t$-measurable.

Throughout the course, $\mathcal{F}_0$ is assumed trivial.

Definition 1.6. Given a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ in discrete time, a process $X = (X_t)_{t \geq 1}$ is predictable if and only if X_t is $\mathcal{F}_{t-1}$-measurable.

Sometimes we need X_0 to be defined, so we just ask that X_0 is $\mathcal{F}_0$-measurable.

Definition 1.7. Given $P = (P_t)_{t \geq 0}$ prices process in discrete time. An investment/consumption strategy is a predictable process (H, c) where H_t takes values in R^n and $c_t \geq 0$ and satisfies the **self-financing condition**

$$H_{t-1} - P_{t-1} = H_t \cdot P_t + c_t \tag{1.1}$$

for all $t \geq 1$.

H_t models the portfolio during $(t-1, t]$, and c_t models the consumption during $(t-1, t]$.

Notation. $X_t(H) = H_t \cdot P_t$ *is the wealth at time t. Note that given H, we can find C by solving the self-financing condition.*

If $c_t = 0$ a.s. for all t then H is a pure investment strategy.

Example 1.8. *Given an initial wealth* $x > 0$*, find* (H, c) *to maximize*

$$\sum_{i=1}^{T} \mathbb{E}(U(c_t))) \tag{1.2}$$

subject to $X_T(H) = 0$ *where* $T > 0$ *is not random.*

Assume that U *is strictly increasing, strongly concave, and bounded from above.*

1.3 A Detour into Martingales

Proposition 1.9. *Let* X *be integrable and* $\mathcal{G} \subseteq \mathcal{F}$*. Then there exists an integrable,* $\mathcal{G}$*-measurable random variable* $\bar{X}$ *such that*

$$\mathbb{E}(X\mathbb{I}(G)) = \mathbb{E}(\bar{X}\mathbb{I}(G))) \tag{1.3}$$

for all $G \in \mathcal{G}$*. Moreover, it is unique in the sense that if* $\bar{\bar{X}}$ *has the same property, then* $\bar{X} = \bar{\bar{X}}$*.*

Definition 1.10. Such $\bar{X}$ is written $\mathbb{E}(X|\mathcal{G})$, the conditional expectation of X given $\mathcal{G}$.

Useful properties of conditional expectation:

(i) If X is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$.

(ii) If X is independent of $\mathcal{G}$ (that is, X and $\mathbb{I}(G)$ are independent for all $G \in \mathcal{G}$), then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$.

(iii) (Tower property) If $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$, then

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})\,|\mathcal{H}) = \mathbb{E}(\mathbb{E}(X|\mathcal{H})\,|\mathcal{G}) = \mathbb{E}(X|\mathcal{H}) \tag{1.4}$$

(iv) (Slot property) If Y is $\mathcal{G}$-measurable and XY is integrable, then

$$\mathbb{E}(XY|\mathcal{G}) = Y\mathbb{E}(X|\mathcal{G}) \tag{1.5}$$

Definition 1.11. A martingale $(X_t)_{t\geq 0}$ with respect to a filtration $\mathbb{F}$ has the properties

- $\mathbb{E}(|X_t|) < \infty$ for all t,
- $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ for all $0 \leq s \leq t$.

Note that X is automatically adapted.

Exercise 1.12. *Suppose* X *is an integrable discrete-time process such that* $\mathbb{E}(X_t|\mathcal{F}_{t-1}) = X_{t-1}$ *for all* $t \geq 1$*. Show that* X *is a martingale.*

Example 1.13. *Let ξ_i, $i = 1, 2, \ldots$ be independent, integrable random variables with $\mathbb{E}(\xi_i) = 0$. Let $\mathcal{F}_t = \sigma(\xi_1, \ldots, \xi_t)$, $X_t = \xi_1 + \xi_2 + \cdots + \xi_t$.*

Then X is a martingale.

Example 1.14. *Let ξ be integrable and let $\mathbb{F}$ be a filtration, and $X_t = \mathbb{E}(\xi | \mathcal{F}_t)$*

Proof. Integrability comes from integrability of conditional expectations.

$$\begin{aligned}
\mathbb{E}(X_t|\mathcal{F}_s) &= \mathbb{E}(\mathbb{E}(\xi|\mathcal{F}_t)\,|\mathcal{F}_s) \\
&= \mathbb{E}(\xi|\mathcal{F}_s) \\
&= X_s
\end{aligned}$$

□

Example 1.15. *Suppose X is a discrete-time martingale and Y is predictable and bounded. Let $Z_t = \sum_{s=1}^{t} Y_s(X_s - X_{s-1})$. Then Z is a martingale.*

Proof. Integrability checked by integrability of X and boundedness of Y.

Z_{t-1} is $\mathcal{F}_{t-1}$ measurable since measurability respects algebraic operations.

$$\begin{aligned}
\mathbb{E}(Z_t|\mathcal{F}_{t-1}) &= \mathbb{E}(Z_{t-1} + Y_t(X_t - X_{t-1})|\mathcal{F}_{t-1}) \\
&= Z_{t-1} + \underbrace{Y_t}_{\text{slot property}} \mathbb{E}\left(\underbrace{X_t - X_{t-1}}_{=0}|\mathcal{F}_{t-1}\right)
\end{aligned}$$

□

Theorem 1.16. *Suppose $u : [0, \infty) \to \mathbb{R}$ is strictly increasing, strictly concave, differentiable, bounded from above. Suppose there exists investment strategy $H^\star$ and consumption $c_t^\star = (H_{t-1}^\star - H_t^\star) \cdot P_{t-1}$, and a state price density $Y^\star$ such that $u'(c_t^\star) = Y_{t-1}^\star$. Then $(H^\star, c^\star)$ is optimal for the problem $\max \sum_{t=1}^{T} \mathbb{E}(u(c_t))$, subject to $X_0(H) = x$, $X_T(H) = 0$.*

Proof. We consider the case where Ω is finite.

Let $L(H, c, Y) = \mathbb{E}\Big(\sum_{t=1}^{T}\left(u(c_t) + Y_{t+1}(H_{t+1}P(t+1) - c_t - H_t \cdot P_{t-1})\right)\Big)$

Note that $L(H, c, Y)$ is the objective when (H, c) is feasible. Then

$$L(H, c, Y) = \mathbb{E}\left(\sum_{t=1}^{T}(u(c_t) - c_t Y_{t-1})\right) + Y_0 X - Y_{t-1} H_t P_{t-1} \\ + \sum_{t=1}^{T-1} H_t(Y_t P_t - Y_{t-1} P_{t-1}) \quad (1.6)$$

First note that $u(c_t^\star) - Y_{t-1}^\star c_t^\star \geq u(c_t) - Y_{t-1}^\star c_t$ since $u'(c_t^\star) = Y_{t-1}^\star$ (first order condition for the maximum of the concave function $c \mapsto u(c) - yc$).

Second, by definition, YP is a martingale, and by finiteness of Ω, the predictable process H is bounded. Therefore, $M_t = \sum_{s=1}^{t} H_s(Y_s P_s - Y_{s-1} P_{s-1})$ is a martingale and $E(M_t) = M_s = 0$.

Putting this together, $L(H, c, Y^\star) \leq L(H^\star, c^\star, Y^\star)$. $\square$

Theorem 1.17. *An absolute arbitrage is an investment/consumption strategy (H, c) such that $X_0(H) = 0, X_T(H) = 0$, at some non-random time horizon $T > 0$, and $\mathbb{P}\left(\sum_{t=1}^{T} c_t > 0\right) > 0$.*

Definition 1.18. A numeraire asset is one whose price is strictly positive almost surely.

Example 1.19. *Here is a market without a numeraire. $P_0 = 1, P_0 = -1, P_2 = 1$.*

Arbitrage:

$$H_1 = -1, c_1 = 1 X_1 = 1, c_2 = 1, H_2 = 0 X_2 = 0$$

Exercise 1.20. *Suppose H_1 is an arbitrage and the market has a numeraire. Then there exists a pure investment strategy H' and a time horizon T' such that $X_0(H') = 0, X_{T'}(H') \geq 0$ a.s., and $\mathbb{P}(X_{T'}(H') > 0) > 0$.*

Theorem 1.21. *A market model has no arbitrage if and only if there exists a state price density.*

Proof. $T = 1$ case. Suppose there exists a state price density $(Y_t)_{t=0,1}$ without loss $Y_0 = 1$. Let $Y = Y_1$ for clarity, $Y > 0$ a.s.

Suppose $(H_t)_{t=1} = H_1 = H$ (non-random vector) is a candidate arbitrage, so $H \cdot P_0 \leq 0$ and $H \cdot P_1 \geq 0$ a.s. We must show $H \cdot P_0 = 0 = H \cdot P_1$ a.s.

Since $Y > 0$, $H \cdot P_1 \geq 0 \Rightarrow \mathbb{E}(YHP_1) \geq 0$, but $H \underbrace{\mathbb{E}(YP_1)}_{\text{state price density}} = HP_0 \leq 0$.

By the pigeonhole principle, if $Z \geq 0$ a.s and $E(Z) = 0$, then $Z = 0$ a.s.

Thus, $YH \cdot P_1 = 0$ a.s., and since $Y > 0$ a.s., $H_0 P_1 = 0 = HP_0 = 0$ a.s.

Now consider the other direction. Let $\mathcal{Y} = \{Y > 0 a.s, \mathbb{E}(Y\|P_1\|) < a\}$. $\mathcal{Y}$ is non-empty since $Y_0 = e^{-\|P_1\|} \in \mathcal{Y}$ and $\mathcal{Y}$ is convex. Let $\mathcal{C} = \{\mathbb{E}(YP_1), y \in \mathcal{Y}\}$. Suppose $P_0 \notin \mathcal{C}$.

By the separating hyperplane theorem, there exists $H \in \mathbb{R}^n$ such that

(i) For all $c \in \mathcal{C}$, $H(c - P_0) \geq 0$.

(ii) There exists $c^\star \in \mathcal{C}$, $H(c^\star - P_0) > 0$.

This implies

(i) For all $Y \in \mathcal{Y}$, $\mathbb{E}(YH \cdot P_1) \geq H \cdot P_0$

(ii) There exists $Y^\star \in \mathcal{Y}$, $\mathbb{E}(YH \cdot P_1) > H \cdot P_0$.

Let $y = \{Y > 0 : \mathbb{E}(Y\|P_1\|) \infty\}$. Let $\mathcal{P} = \{\mathbb{E}(YP_1) : Y \in \mathcal{Y}\} \subseteq \mathbb{R}^n$. Suppose $P_0 \notin \mathcal{P}$.

By the **separating/supporting hyperplane theorem** there exists a vector $H \in \mathbb{R}^n$ such that

(i) For all $p \in \mathcal{P}$, $H \cdot (p - P_0) \geq 0$,

(ii) There exists $p^\star \in \mathcal{P}$ such that $H \cdot (p^\star - P_0) > 0$.

If $p \in \mathcal{P}$ then $p = \mathbb{E}(YP_1)$ for some Y. Then

$$H \cdot p = \mathbb{E}\left(Y \underbrace{H \cdot P_1}_{X\text{, time 1 wealth}}\right), H \cdot P_0 = \underbrace{-c}_{\text{consumption in } (0,1]} \tag{1.7}$$

Restating, we then have:

(i) For all $Y \in \mathcal{Y}$, $\mathbb{E}(YH \cdot P_1) \geq H \cdot P_0$

(ii) There exists $Y^\star \in \mathcal{Y}$, $\mathbb{E}(YH \cdot P_1) > H \cdot P_0$.

We need to show that $X \geq 0$ a.s., $c \geq 0$, $\mathbb{P}(X+c>0) > 0$. Let $Y_0 = e^{-\|P_0\|} \in \mathcal{Y}$. For $\epsilon > 0$, let $Y = \epsilon Y_0$ in (i), then $\epsilon \mathbb{E}(Y_0 X) \geq c \Rightarrow c \geq 0$ by taking $\epsilon \to 0$.

Let $Y = (\frac{1}{\epsilon}\mathbb{I}(X<0)+1)Y_0$ in (i), which implies

$$\mathbb{E}(Y_0 X \mathbb{I}(X<0)) \geq -\epsilon(\mathbb{E}(X_0 Y)+c) \to 0 \tag{1.8}$$

as $\epsilon \to 0$.

Then $Y_0 > 0$, $X\mathbb{I}(X<0) \leq 0$ by pigeonhole principle,

$$\mathbb{P}(X<0) = 0 \Rightarrow X \geq 0 \tag{1.9}$$

a.s.

By (ii), $\mathbb{P}(X=0, c=0) < 1$. □

Definition 1.22. An integrable adapted process X is a supermartingale is a supermartingale if

$$\mathbb{E}(X_t|\mathcal{F}_s) \leq X_s \tag{1.10}$$

for all $0 \leq s \leq t$.

Proposition 1.23. *If X is a supermartingale and $\mathbb{E}(X_T) = X_0$ for some non-random $T > 0$, then $(X_t)_{0\leq t\leq T}$ is a martingale.*

Proof. Let $Y_{s,t} = X_s - \mathbb{E}(X_t|\mathcal{F}_s) \geq 0$ by assumption. Then

$$\begin{aligned}
\mathbb{E}(Y_{s,t}) &= \mathbb{E}(X_s - \mathbb{E}(\mathbb{E}(X_T|\mathcal{F}_s))) \\
&= \mathbb{E}(X_s) - \mathbb{E}(X_T) \\
&\leq \underbrace{X_0}_{\text{supermartingale}} - \underbrace{X_0}_{\text{by assumption}}
\end{aligned}$$

By pigeonhole, $Y_{s,T} = 0$ a.s. Then $X_s = \mathbb{E}(X_T|\mathcal{F}_s)$ for all $0 \leq s \leq T$. So by the tower property, $(X_s)_{0\leq s\leq T}$ is a martingale. □

Proof (Easy direction of 1FTAP). Let $T > 1$, and finite sample space. Let H be a strategy, and $X = X(H)$ be a corresponding wealth process. Let Y be a state price density. Then XY is a supermartingale, as[1]

[1] This relies on the finiteness of Ω since this guarantees that H is bounded, and so we call use the slot property

$$
\begin{aligned}
\mathbb{E}(X_t Y_t | \mathcal{F}_{t-1}) &= \mathbb{E}(H_t \cdot P_t Y_t | \mathcal{F}_{t-1}) \\
&= \underbrace{H_t}_{\text{slot property}} \cdot \mathbb{E}(P_t Y_t | \mathcal{F}_{t-1}) \\
&= H_t \cdot P_{t-1} Y_{t-1} \\
&= (H_{t-1} P_{t-1} - c_t) Y_{t-1} \\
&\leq X_{t-1} Y_{t-1}.
\end{aligned}
$$

Suppose H is such that $X_0 = 0$ and $X_T = 0$ a.s. for some non-random $T > 0$. Then

$$
\mathbb{E}(Y_T X_T) = 0 = Y_0 X_0 \tag{1.11}
$$

and so XY is a martingale by the previous proposition. This implies $Y_t X_t = \mathbb{E}(Y_t X_t | \mathcal{F}_t) = 0$, which implies $X_t = 0$ for all t.

By the calculation,

$$
\begin{aligned}
\mathbb{E}(X_t Y_t | \mathcal{F}_{t-1}) &= (X_{t-1} + c_t) Y_{t-1} \\
&\Rightarrow c_t = 0
\end{aligned}
$$

for all t. □

Definition 1.24. A stopping time for a filtration $(F_t)_{t \in \mathbb{T}}$ is a random variable $\tau : \Omega \to \mathbb{T} \cup \{\infty\}$ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in \mathbb{T}$ (discrete or continuous time).

Notation. *$M_{t \wedge \tau} = M_t^\tau$ is the martingale M stopped at τ.*

Proposition 1.25. *Let M be a martingale and τ a stopping time, and let $N_t = M_{t \wedge \tau}$. Then N is also a martingale.*

Proof.

$$
N_t = M_0 + \sum_{s=1}^{t} \mathbb{I}(s \leq \tau) \, (M_s - M_{s-1}) \tag{1.12}
$$

and $\mathbb{I}(\tau \leq s-1)$ is $\mathcal{F}_{s-1}$-measurable and bounded. □

Definition 1.26. A local martingale is an adapted process X such that there exists an increasing sequence of stopping times $\tau_n \uparrow \infty$ such that X^{τ_n} is a martingale for all n.

Remark 1.27. *Martingales are local martingales.*

Proposition 1.28. *Let X be a local martingale (discrete time). Let K be predictable and let $Y_t = \sum_{s=1}^t K_s(X_s - X_{s-1})$. Then Y is a local martingale.*

Proof. Since X is a local martingale, there exists a sequence $\sigma_n \to \infty$ stopping times such that X^{σ_n} is a martingale. Let

$$\tau_n = \inf\{t \geq 0 : |K_{t+1}| > N\} \tag{1.13}$$

Then we have

$$X_{t\wedge(\underbrace{\sigma_n \wedge \tau_n}_{\text{stopping time}})} = \sum_{s=1}^{t} \underbrace{K_s \mathbb{I}(s \leq \tau_n)}_{\text{bounded and predictable}} (\underbrace{X_s^{\tau_n} - X_{s-1}^{\tau_n}}_{\text{martingale difference}}) \tag{1.14}$$

□

Example 1.29. *Let ν, ξ be random variables with ξ integrable and $\mathbb{E}(\xi) = 0$. Let $\mathcal{F}_1 = \sigma(\nu), \mathcal{F}_2 = \sigma(\nu, \xi)$. Let $X_1 = 0$, $X_2 = \nu\xi$. Then X is a local martingale.*

If the product $\nu\xi$ is also integrable, then X is a true martingale, otherwise $\mathbb{E}(X_2|\mathcal{F}_1)$ is not defined.

Proposition 1.30. *Let X be a local martingale such that there exists an integrable process Y such that $Y_t \geq |X_s|$ for all $0 \leq s \leq t$. Then X is a true martingale.*

Proof. By assumptions there exists a sequence $\tau_N \to \infty$ such that X^{τ_N} is a martingale. Also, $|X_{t\wedge\tau_N} \leq Y_t$ which is integrable. Then

$$\begin{aligned}
\mathbb{E}(X_t|\mathcal{F}_s) &= \mathbb{E}\left(\lim_{N\to\infty} X_{t\wedge\tau_N}|\mathcal{F}_s\right) && (1.15)\\
&= \lim_{N\to\infty} \mathbb{E}(X_{t\wedge\tau_N}|\mathcal{F}_s) && (1.16)\\
&= \lim_{N\to\infty} X_{s\wedge\tau_N} && (1.17)\\
&= X_s && (1.18)
\end{aligned}$$

□

Corollary 1.31. ***In discrete time**, if X is a local martingale and $\mathbb{E}(|X_t|) < \infty$ for all $t \geq 0$ then X is a martingale.*

Proof. Let $Y_t = \sum_{s=0}^t |X_s|$, and Y is integrable by assumption. $\square$

Proposition 1.32. *If X is a local martingale (in discrete or continuous time) and $X_t \geq 0$ almost surely for all t, then X is a supermartingale.*

Proof. First, X_t is integrable, since

$$\begin{aligned}
\mathbb{E}(|X_t|) &= \mathbb{E}(X_t) && (1.19)\\
&= \mathbb{E}\left(\lim_{N\to\infty} X_{t\wedge\tau_N}\right) && (1.20)\\
&\leq \liminf_{N\to\infty} \mathbb{E}(X_{t\wedge\tau_N}) && (1.21)\\
&= \liminf_{N\to\infty} X_{0\wedge\tau_n} && (1.22)\\
&= X_0 < \infty. && (1.23)
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbb{E}(X_t|\mathcal{F}_s) &= \mathbb{E}(\lim X_{t\wedge\tau_N}|\mathcal{F}_s) && (1.24)\\
&\leq \liminf \mathbb{E}(X_{t\wedge\tau_N}|\mathcal{F}_s) && (1.25)\\
&= \liminf X_{s\wedge\tau_N} && (1.26)\\
&= X_s && (1.27)
\end{aligned}$$

$\square$

Corollary 1.33. ***In discrete time**, non-negative local martingales in discrete time are martingales.*

Proof. Let X be the local martingale. Then $\mathbb{E}(|X_t|) < \infty$ for all $t \geq 0$ by Fatau. The result follows from the last corollary. $\square$

Theorem 1.34. *Let X be a **discrete** time local martingale. Fix $T > 0$ non-random. Then $(X_t)_{0\leq t\leq T}$ is a true martingale if either*

(i) $\mathbb{E}(|X_T|) < \infty$, *or*

(ii) $X_T \geq 0$

Lecture on Wednesday 23 October

1.4 *Contingent Claims*

Setup - P is a price process (n-dimensional space, adapted).

Two types of claims

(i) European - specified by a time horizon T (maturity date or expiry) and a $\mathcal{F}_T$-measurable random variable ξ_T (the payout of the claim).

(ii) American - specified maturity date T and an adapted process $(\xi_t)_{0\leq t\leq T}$ where ξ_t is the payout if owner of claim chooses to exercise at time $t \leq T$.

Example 1.35. *A call option is the right, but not the obligation, to buy a certain stock at a fixed price sometime in the future.*

$$\xi_T = (S_T - k)^+ \tag{1.28}$$

$$\xi_t = (S_t - k)^+ \tag{1.29}$$

for all $0 \leq t \leq T$*.*

Definition 1.36. A European contingent claim is **attainable** or **replicable** if there exists a pure investment strategy H such that $X_T(H) = \xi_T$ almost surely.

Theorem 1.37. *Suppose* ξ_t *is the price of attainable claim for* $0 \leq t \leq T$*. If the augmented market* (P, ξ) *has no arbitrage then* $\xi_t = X_t(H)$ *a.s.*

Proof. Let $\tau = \inf\{t \geq 0 : X_t \neq \xi_t\}$. Let $\bar{H}_t = \text{sign}(\xi_t, X_t)\mathbb{I}(t \geq \tau)\,(H_t, -1)$.

Then $c_{\tau+1} = |\xi_\tau - X_\tau|$, $\bar{X}_t(\bar{H}) = \bar{H}_t \cdot (P_t, \xi_t)$, $\bar{X}_0(\bar{H}) = 0, \bar{X}_T(\bar{H}) = 0$, and $c_t = 0$ for all t if and only if there is no arbitrage. $\square$

Theorem 1.38. *Suppose* Y *is a state price density of the original market with prices* P*. Suppose* ξ_T *is the payout of an attainable claim, suppose either*

(i) $\mathbb{E}(|\xi_T| Y_T) < \infty$*, or*

(ii) $\xi_T \geq 0$ *a.s.*

If the augmented market (P, ξ) *has no arbitrage, then*

$$\xi_t = \frac{1}{Y_t}\mathbb{E}(Y_T \xi_T | \mathcal{F}_t) \tag{1.30}$$

for all $0 \leq t \leq T$*.*

Proof. By the previous result, there exists H (pure investment strategy) such that $X_t(H) = \xi_t$ for all t. But XY is a local martingale. From before, if either $X_T Y_T$ is integrable or non-negative, the process XY is a true martingale.

$$\xi_t Y_t = X_t Y_t = \mathbb{E}(X_T Y_T | \mathcal{F}_t) = \mathbb{E}(\xi_T Y_T | \mathcal{F}_t) \tag{1.31}$$

as required. □

Remark 1.39. *When our price process can be decomposed into a numeraire, so $P = (N, S)$, we can let $\mathbb{Q}$ be an equivalent martingale measure. If either $\mathbb{E}_{\mathbb{Q}}\left(\frac{\xi_T}{N_T}\right) < \infty$, or $\xi_T \geq 0$, then*

$$\xi_t = N_t \mathbb{E}_{\mathbb{Q}}\left(\frac{\xi_T}{N_T} | \mathcal{F}_t\right) \tag{1.32}$$

Theorem 1.40. *Suppose ξ_t is the price of a contingent claim at time t (not necessarily attainable). Suppose that the augmented market (P, ξ) has no arbitrage. Then there exists a positive process Y such that*

$$P_t = \frac{1}{Y_t} \mathbb{E}(Y_T P_T | \mathcal{F}_t) \tag{1.33}$$

$$\xi_t = \frac{1}{Y_t} \mathbb{E}(Y_T \xi_T | \mathcal{F}_t) \tag{1.34}$$

Here, (1.33) shows Y is a state price density for the original market, and (1.34) shows Y is a state price density for the augmented market.

Proof. The proof is just 1FTAP applied to the augmented market. □

Example 1.41. *Let $P_t = (B_{t,T}, S_t)$. $B_{t,T}$ is price of bond maturing at T, with $B_{T,T} = 1$ almost surely. S_t is a stock with $S_t \geq 0$ for all t. Let c_t be the price of a call with payout $(S_T - K)^+$. Suppose $(B_{t,T}, S_t, C_t)_{t \in [0,T]}$ has no arbitrage.*

In general, since the payout of the call is non-negative then $c_t \geq 0$. Also, $(S_T - K)^+ \geq S_T - K = S_T - KB_{T,T} = (-K, 1) \cdot (B_{t,T}, S_t)$.

This implies

$$c_t \geq S_t - KB_{t,T} \tag{1.35}$$

Then $c_t \geq (S_t - KB_{t,T})^+$, and $(S_T - K)^+ < S_T$, thus $c_t \leq S_t$.

If there exists a state price density Y for (B, S) such that

$$c_t = \frac{1}{Y_t}\mathbb{E}\big(Y_T(S_T - K)^+|\mathcal{F}_t\big)\,. \tag{1.36}$$

Example 1.42. *A put option is equivalent to $(K - S_T)^+ = K - S_T + (S_T - K)^+ = (K, -1, 1) \cdot (B_{T,T}, S_T, C_T)$. If p_t is a no-arbitrage price of the put, then*

$$p_t = KB_{t,T} - S_t + c_t. \tag{1.37}$$

Definition 1.43. A market is **complete** if and only if every European contingent claim is attainable. A market that is not complete is **incomplete**.

Theorem 1.44 (Second fundamental theorem of asset pricing)**.** *A market with no arbitrage is complete if and only if there exists a unique (up to scaling) state price density.*

Proof. Suppose the market is complete. Let Y, Y' be state price densities with $Y_0 = Y_0' = 1$. Fix $T > 0$ and let $\xi_T \geq 0$ be $\mathcal{F}_T$-measurable. By completeness, there exists a pure investment strategy H such that $X_T(H) = \xi_T$.

From before,

$$\mathbb{E}(Y_T\xi_T) = X_0(H) = \mathbb{E}\big(Y_T'\xi_T\big) \tag{1.38}$$

and thus $\mathbb{E}(\xi_T(Y_T - Y_T')) = 0$. Let $\xi_T = \mathbb{I}(Y_T > Y_T')$. Then $Y_T \leq Y_T'$ almost surely, and so by symmetry, $Y_T = Y_T'$.

A claim with payout $\xi_T \geq 0$ is attainable if there exists $x \geq 0$ such that $\mathbb{E}\left(\frac{Y_T\xi_T}{Y_0}\right) = x = X_0(H)$ for all state price densities.[2]

[2] Proof in example sheet

Given there exists a unique state price density, every non-negative claim is attainable. The conclusion follows by observing $\xi_T = \xi_T^+ - \xi_T^-$. □

Theorem 1.45. *Suppose that the price process P is n-dimensional and the market is complete. The for each $t \geq 0$, there are no more than n^t disjoint sets of positive probability $\mathcal{F}_t$-measurable sets of positive probability. In particular, the random vector P_t takes on at most n^tvalues.*

Proof. Consider the $t = 1$ case. Let $A_1, \ldots, A_k$ be disjoint $\mathcal{F}_1$-measurable sets with $\mathbb{P}(A_i) > 0$. We claim the set $\{\mathbb{I}(A_i)\}$ is linearly independent.

Suppose $\sum_i a_i \mathbb{I}(A_i) = 0$. Multiplying by $\mathbb{I}(A_j)$ implies $a_j \mathbb{I}(A_j) = 0$ almost surely by disjointness. Since $\mathbb{P}(A_j) > 0$ by assumption we have $a_j = 0$.

By completeness, each $\mathbb{I}(A_i)$ is attainable, so

$$\operatorname{span}\{\mathbb{I}(A_i)\} \subseteq \{H \cdot P_1, H \in \mathbb{R}^n\} = \operatorname{span}\{P_1^1, \ldots, P_1^n\} \tag{1.39}$$

□

1.5 *American Claims*

Recall that the payoff of an American claim is specified by an adapted process $(\xi_t)_{0 \le t \le T}$ where ξ_t is the payout if the claim is executed at time t.

Theorem 1.46. *Suppose the market is complete. Then there exists a (pure investment) strategy such that $X_t(H) \ge \xi_t$ for all $0 \le t \le T$, and there exists a stopping time $\tau^\star$ such that $X_{\tau^\star}(H) = \xi_{\tau^\star}$.*

Furthermore, $X_0(H) = \sup_{\text{stopping time } \tau \le T} \mathbb{E}(Y_\tau \xi_\tau)$ where Y is the unique state price density such that $Y_0 = 1$.

Definition 1.47. Let Z be an adapted integrable process $(Z_t)_{0 \le t \le T}$. The Snell envelope of $Z$is the process U defined by $U_T = Z_T$, $U_t = \max\{Z_t, \mathbb{E}(U_{t+1} | \mathcal{F}_t)\}$ for $0 \le t \le T - 1$.

Remark 1.48. *Note that $U_t \ge Z_t$for all t, and U is a supermartingale since $U_t \ge \mathbb{E}(U_{t+1} | \mathcal{F}_t)$.*

Theorem 1.49 (Doob decomposition)**.** *Let U be a discrete-time supermartingale. Then there exists a martingale M with $M_0 = 0$, and a non-decreasing process A with $A_0 = 0$ such that $U_t = U_0 + M_t - A_t$.*

Proof. Let $M_0 = A_0 = 0$, $M_{t+1} = M_t + U_{t+1} - \mathbb{E}(U_{t+1} | \mathcal{F}_t)$, and $A_{t+1} = A_t + U_t - \mathbb{E}(U_{t+1} | \mathcal{F}_t)$. By induction, A_t is predictable. A is non-decreasing as U is a supermartingale.

Now, we show uniqueness. Suppose $U = U_0 + M - A = U_0 + M' - A'$. Then $M - M' = A - A'$, and as $A - A'$ is predictable, we have $M - M'$ is a predictable martingale. In discrete time, predictable martingales are almost surely constant. Thus, $M_t - M'_t = M_0 - M'_0 = 0$, and thus we have demonstrated uniqueness. □

Theorem 1.50. *Let Z be integrable and adapted, U is a Snell envelope, with Doob decomposition $U = U_0 + M - A$. Let $\tau^\star = \inf\{t \geq 0 | A_{t+1} > 0\}$ with the convention $\tau^\star = T$ on $\{A_t = 0 \forall t\}$.*

Then $U_{\tau^\star} = U_0 + M_{\tau^\star} = Z_{\tau^\star}$.

Remark 1.51. *$\tau^\star$ is a stopping time since A is predictable.*

Proof. Note that $A_{\tau^\star} = 0$ but $A_{\tau^\star+1} > 0$. We have

$$\begin{aligned} U_t &= U_0 + M_t - A_t && (1.40) \\ &= \max\{Z_t, \mathbb{E}(U_{t+1}|\mathcal{F}_t)\} && (1.41) \\ &= \max\{Z_t, U_0 + M_t - A_{t+1}\}. && (1.42) \end{aligned}$$

So $U_0 + M_{\tau^\star} = \max\{Z_{\tau^\star}, U_0 + M_{\tau^\star} - A_{\tau^\star - 1}\}$, which implies $U_0 + M_{\tau^\star} = Z_{\tau^\star} = U_{\tau^\star}$ as required. □

Theorem 1.52. *Under the same hypothesis as before,*

$$U_0 = \sup_{\text{stopping times } \tau \leq T} \mathbb{E}(Z_\tau) \,. \tag{1.43}$$

Proof. By the optional stopping theorem, $U_0 \geq \mathbb{E}(U_\tau) \leq \mathbb{E}(Z_t)$ for any stopping time $\tau \leq T$, and since $U_t \geq Z_t \forall t$.

But $U_0 = \mathbb{E}(U_0 + M_{\tau^\star}) = \mathbb{E}(Z_{\tau^\star})$. □

We now give a proof of the existence of the minimal super-replicating strategy. Let U be the Snell envelope of $(Y_t \xi_t)_{0 \leq t \leq T}$. Let $U = U_0 + M - A$ be its Doob decomposition.

By completeness, there exists a strategy H such that

$$X_T(H) = \frac{U_0 + M_T}{Y_T}. \tag{1.44}$$

Since XY is a martingale (XY is a local martingale in general but by completeness all processes are bounded). So

$$
\begin{aligned}
X_t Y_T &= U_0 + M_t && (1.45)\\
&\geq U_0 + M_t - A_t && (1.46)\\
&= U_t && (1.47)\\
&\geq Y_t \xi_t. && (1.48)
\end{aligned}
$$

Thus $X_t \geq \xi_t$ for all $0 \leq t \leq T$.

Also, at $\tau^\star = \inf\{t \geq 0 | A_{t+1} > 0\}$, we have

$$X_{\tau^\star} Y_{\tau^\star} = U_0 + M_{\tau^\star} = U_{\tau^\star} = Y_{\tau^\star} \xi_{\tau^\star}, \tag{1.49}$$

and so $X_{\tau^\star} = \xi_{\tau^\star}$.

Note also that $X_0 = \mathbb{E}(U_0 + M_T) = U_0 = \sup_{\tau \leq T} \mathbb{E}(\xi_\tau Y_\tau)$.

2

Continuous Time Models

In discrete time, we had $X_t - X_{t-1} = H_t \cdot (P_t - P_{t-1}) - c_t$. For continuous time, we replace this with $dX_t = H_t dP_t - c_t dt$

A state price density is some stochastic process Y with $Y_t > 0$ and YP is a martingale

Lemma 2.1. *If $t \mapsto X_t(\omega)$ is differentiable and X is a martingale then X is constant.*

This can make a pricing theory quite boring!

2.1 Diversion into Stochastic Calculus

Definition 2.2. A (standard scalar) Brownian motion is a process $W = (W_t)_{t \geq 0}$ such that

(i) $W_0(\omega) = 0$ for all ω.

(ii) $t \mapsto W_t(w)$ is continuous for all ω

(iii) For any $0 \leq t_0 < t_1 < \cdots < t_n$, the increments $W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent, with $W_t - W_s \sim N(0, |t-s|)$.

Theorem 2.3. *The Brownian motion exists (Weiner, 1923).*

Consider a filtration $(\mathcal{F}_t)$ with the property that $W_t - W_s$ is independent of $\mathcal{F}_s$, $0 \leq s \leq t$. Our technical assumptions are usual conditions - $\mathcal{F}_t = \cap_{\epsilon>0} \mathcal{F}_{t+\epsilon}$ (right-continuity), $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.

Definition 2.4. A **simple predictable process** is of the form

$$\alpha_t(\omega) = \sum_{i=1}^{n} \mathbb{I}((t_{i-1}, t_i))\, a_i(\omega), \tag{2.1}$$

where $0 \leq t_0 < \cdots < t_n$, each a_i is a bounded $\mathcal{F}_{t_{i-1}}$-measurable random variable.

Remark 2.5. *α is left-continuous, piecewise-constant, and adapted.*

Definition 2.6.

$$\int_0^\infty \alpha_s dW_s = \sum_{i=1}^{n} a_i(W_{t_i} - W_{t_{i-1}}) \tag{2.2}$$

where α is a simple predictable process.

Definition 2.7. The predictable σ-algebra on $[0, \infty) \times \Omega$ is generated by $(s, t] \times A$ where $A \in \mathcal{F}_s$.

This is the smallest σ-algebra for which simple predictable processes are measurable.

A process measurable with respect to the predictable σ-algebra is called **predictable**.

Remark 2.8. *If α is left-continuous and adapted, it is predictable.*

Proposition 2.9 (Itô's isometry)**.** *If α is simple and predictable, then*

$$\mathbb{E}\left(\left(\int_0^\infty \alpha_s dW_s\right)^2\right) = \mathbb{E}\left(\int_0^\infty \alpha_s^2 ds\right) \tag{2.3}$$

Thus, the isometry I from simple predictable process to square integrable random variables on $L^2(\Omega, \mathcal{F}, \mathbb{P})$ (which is complete) defined by

$$I(\alpha) = \int_0^\infty \alpha_s dW_s \tag{2.4}$$

Proof.

$$\left(\int \alpha dW\right)^2 = \left(\sum a_i \Delta W_i\right)^2 \tag{2.5}$$

$$= 2\sum_{j<i} a_j a_i \Delta W_j \Delta W_i + \sum a_i^2 (\Delta W_i)^2 \tag{2.6}$$

Note that $\mathbb{E}\left(\sum a_i^{2(\Delta W_i)^2}\right) = ...$

□ Finish this proof

Definition 2.10. Suppose $\mathbb{E}\left(\int_0^\infty (\alpha_s^k - \alpha_s)^2 ds\right) \to 0$, where each α^k is simple and predictable. Then

$$\int_0^\infty \alpha_s dW_s = \lim_{L^2} \int_0^\infty a_s^k dW_s \tag{2.7}$$

Theorem 2.11. *If α is predictable and $\mathbb{E}\left(\int_0^t \alpha_s^2 ds\right) < \infty$ for all t, there exists a continuous martingale X such that $X_t = \int_0^\infty \alpha_s \mathbb{I}(s \leq t)\, dW_s$.*

For notation, we represent X_t as $\int_0^t \alpha_s dW_s$. Note that $\mathbb{E}(X_t) = 0$ and $\mathbb{E}(X_t^2) = \int_0^t \alpha_s^2 ds$.

Definition 2.12 (Localization)**.** Suppose α is predictable and $\int_0^t \alpha_s^2 ds < \infty$ almost surely for all t. Let $\tau_n = \inf\{t \geq 0 | \int_0^t \alpha_s ds > n\}$.

Let $\alpha_t^{(n)} = \alpha_t \mathbb{I}(t \leq \tau_n)$, so $\int_0^t \alpha_s^{(n)} dW_s$ is well-defined by the L^2 theory, since $\mathbb{E}\left(\int_0^t (\alpha_s^{(n)})^2 ds\right) \leq N \leq \infty$ as $\int_0^t \alpha_s^2 ds < \infty$ almost surely as $\tau_n \uparrow \infty$.

Notation. *$\int_0^t \alpha_s dW_s$ as $\int_0^t \alpha_s^{(N)} dW_s$ on $\{t \leq \tau_n\}$.*

Theorem 2.13. *If α is adapted and continuous, then $\int_0^t \alpha_s dW_s$ is defined for all $t \geq 0$ - since $t \mapsto \alpha_t(\omega)$ is continuous, α is bounded on $[0, t]$ for each ω, and so $\int_0^t \alpha_s ds < \infty$ almost surely.*

If $X_t = \int_0^t \alpha_s dW_s$, then X is a continuous local martingale, since $X^{(n)} = (X_{t\wedge\tau_n})_t \geq 0$ is a true martingale, where $\tau_n = \inf\{\tau \geq 0, \int_0^t \alpha_s ds \geq N\}$.

2.2 *Itô's Formula*

Definition 2.14. An Itô process X is of the form

$$X_t = X_0 + \int_0^t \alpha_s dW_s + \int_0^t \beta_s ds \tag{2.8}$$

such that α, β are predictable and $\int_0^t \alpha_s ds < \infty$ and $\int_0^t |\beta_s| ds < \infty$ for all t.

Theorem 2.15. *If X is an Itô process and $f \in C^2$, then $f(X)$ is an Itô process. In fact,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\alpha_s dW_s + \int_0^t \left(f'(X_s)\beta_s + \underbrace{\frac{1}{2} f''(X_s)\alpha_s^2}_{\text{Itô's correction}} \right) ds \tag{2.9}$$

Example 2.16. *$f(x) = x^2$. Then*

$$W_t^2 = \int_0^t 2W_s dW_s + t \tag{2.10}$$

$$\mathbb{E}\left(W_t^2\right) = \mathbb{E}\left(\int_0^t 2W_s dW_s\right) + t \tag{2.11}$$

and the first term is zero as it is a martingale.

This follows from

$$\mathbb{E}\left(\int_0^t W_s^2 ds\right) = \int_0^t s ds = \frac{t^2}{2} < \infty \tag{2.12}$$

so $\int_0^t W_s dW_s$ is a martingale.

Theorem 2.17. *Let X be an Itô process. Fix $t > 0$. Then*

$$\lim_{n\to\infty} \sum_{k=1}^n \left(X_{\frac{tk}{n}} - X_{\frac{t(k-1)}{n}}\right)^2 = \int_0^t \alpha_s^2 ds \tag{2.13}$$

Notation.

$$\langle X \rangle_t = \int_0^t \alpha_s ds \tag{2.14}$$

is called the quadratic variation of X.

Theorem 2.18 (Itô's formula)**.** *In integral form,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2}\int_0^t f''(X_s) d\langle X \rangle_s \tag{2.15}$$

In differential form,

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2} f''(X_t) d\langle X \rangle_t \tag{2.16}$$

Morally, the idea is to take Taylor expansion around $f(X_t)$.

Theorem 2.19 (Itô's formula, multidimensional version)**.** *let X, Y be Itô processes. Then the quadratic covariation*

$$\langle X, Y \rangle_t = \lim_{n\to\infty} \sum_{k=1}^n (X_{\frac{tk}{n}} - X_{\frac{t(k-1)}{n}})(Y_{\frac{tk}{n}} - Y_{\frac{t(k-1)}{n}}) \tag{2.17}$$

$$= \frac{1}{2}\langle X + Y \rangle_t - \langle X \rangle_t - \langle Y \rangle_t \tag{2.18}$$

Proposition 2.20. *The quadratic covariance satisfies the following properties:*

(i) *(Bilinear, symmetric)*

$$\langle aX + bY, Z\rangle = a\langle X, Z\rangle + b\langle Y, Z\rangle = \langle Z, aX + bY\rangle \tag{2.19}$$

(ii) *If $X_t = X_0 + \int_0^t \beta_s ds$ then $\langle X, Y\rangle_t = 0$ for any Itô process Y.*

(iii) *Let W^1, W^2 be two independent Brownian motions. Then $\langle W^1, W^2\rangle_t = 0$.*

(iv)
$$\left\langle \int_0^t \alpha_s dW_s, \int_0^t \beta_s dW_s \right\rangle = \int_0^t \alpha_s \beta_s ds \tag{2.20}$$

Let X be an n-dimensional Itô process, and $f \in C^2(\mathbb{R}^n \to \mathbb{R})$. Then

(2.21)

Fill in this multivariate Itô's result

In finance there are state price densities $\Rightarrow$ equivalent martingale measures. How to do computations under equivalent changes of measure?

Let W be an n-dimensional BM with $W = (W^1, \ldots, W^m)$ where W^i are independent standard Brownian motions. Let α be an n-dimensional predictable process and $\int_0^t \|a_s\|^2 ds < \infty$, and let

$$Z_t = e^{\int_0^t \alpha_s dW_s - \frac{1}{2}\int_0^t \|\alpha_s\|^2 ds}. \tag{2.22}$$

Proposition 2.21. *Z satisfies the following properties:*

(i) *Z is a local martingale.*

(ii) *Z is a supermartingale.*

(iii) *If $\mathbb{E}(Z_T) = 1$ for some $T > 0$ (non-random), then $(Z_t)_{0\le t\le T}$ is a true martingale.*

Proof. Let $dX_t = \alpha_t \cdot dW_t - \frac{1}{2}\|\alpha_t\|^2 dt$, $X_0 = 0$. Let $f(x) = e^x$. Then

$$dZ_t = df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X\rangle_t \tag{2.23}$$

Note that

$$d\langle X\rangle_t = d\left\langle \sum_{i=1}^{m} \int_0^t \alpha_s^2 dW_s^2 \right\rangle_t \tag{2.24}$$

$$= d\sum_{i,j} \left\langle \int \alpha_s^i dW_s^i, \int \alpha_s^j dW^j \right\rangle_t \tag{2.25}$$

$$= \sum (\alpha_t^i)^2 dt \tag{2.26}$$

$$= \|\alpha_t\|^2 dt \tag{2.27}$$

Then

$$dZ_t = Z_t \left(\alpha_t \cdot dW_t - \frac{1}{2}\|\alpha_t\|^2 dt \right) + \frac{1}{2} Z_t \|\alpha_t\|^2 dt = Z_t \alpha_t dW_t. \tag{2.28}$$

Thus

$$Z_t = 1 + \int_0^t Z_s \alpha_s \cdot dW_s \tag{2.29}$$

and so Z is a stochastic integral, and hence a local martingale.

$Z_t > 0$ almost surely, so non-negative local martingales are supermartingales by Fatou's lemma.

Z is a supermartingale and $\mathbb{E}(Z_T) = Z_0$, and so $(Z_t)_{0 \le t \le T}$ is a martingale (pigeonhole principle). □

Theorem 2.22 (Cameron-Martin-Girsanov theorem)**.** *Let Zbe as before and assume $\mathbb{E}(Z_T) = 1$for some $T > 0$. Define an equivalent martingale measure $\mathbb{Q}$ by Radon-Nikodym density*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_t \tag{2.30}$$

Let $\hat{W}_t = W_t - \int_0^t \alpha_s ds$. Then $\hat{W}$ is a $\mathbb{Q}$-Brownian motion.

Theorem 2.23 (Martingale representation theorem)**.** *Let W be an m-dimensional Brownian motion generating the filtration $(\mathcal{F}_t)_{t \ge 0}$. Let X be a continuous local martingale. Then there exists a predictable α with $\int_0^t \|\alpha_s\|^2 ds < \infty$ almost surely for all t such that $X_t = X_0 + \int_0^t \alpha_s dW_s$.*

If $X_t > 0$ a.s. for all t, then there exists a predictable process β with $\int_0^t \|\beta_s\|^2 ds < \infty$ for all t such that

$$X_t = X_0 e^{\int_0^t \beta_s dW_s - \frac{1}{2}\int_0^t \|\beta_s\|^2 ds} \tag{2.31}$$

Theorem 2.24 (Levy's characterization theorem)**.** *Let X be a continuous* *local martingale (in any filtration satisfying the usual conditions) such that its quadratic variation $\langle X \rangle_t = t$. Then X is a Brownian motion.*

2.3 Arbitrage Theory in Continuous Time

Recall that in discrete time,

$$X_t = H_t \cdot P_t = H_{t+1} \cdot P_t - c_{t+1} \quad (2.32)$$

$$X_{t+1} = H_{t+1} \cdot P_{t+1} \Rightarrow X_{t+1} - X_t = H_{t+1} \cdot (P_{t+1} - P_t) - c_{t+1} \quad (2.33)$$

The setup is as follows:

(i) P is an m-dimensional Itô process.

Definition 2.25. A self-financing investment/consumption strategy (H, c) is a pair of predictable processes such that $c_t \geq 0$ for all t , $\int_0^t \sum (H_s^i)^2 d\langle P^i \rangle >_s < \infty$ for all t, and

$$H_t \cdot P_t = H_0 \cdot P_0 + \int_0^t H_s \cdot dP_s - \int_0^t c_s ds \quad (2.34)$$

Definition 2.26 (Incomplete)**.** An arbitrage is an investment/consumption strategy (H, c) such that $X_0 = X_T = 0$ and $\mathbb{P}\left(\int_0^T c_s ds > 0\right) > 0$ for some non-random $T > 0$

This definition is flawed.

Example 2.27 (Doubling strategies)**.** *Consider the discrete-time model $P = (1, S_t)$ where $S_t = \xi_1 + \cdots + \xi_t$ where ξ_i are IID with $\mathbb{P}(\xi_i = \pm 1) = \frac{1}{2}$.*

Consider a price vector $P = (1, W)$ with W a Brownian motion . Let $X_t = \int_0^t \pi_s dW_s$, and let $f : [0, 1] \to [0, \infty]$ an increasing bijection with inverse f^{-1}. For example, $f(t) = \frac{t}{1-t}$ with $f^{-1}(u) = \frac{u}{1+u}$. Consider

$$Z_u = \int_0^{f^{-1}(u)} \sqrt{f'(s)} dW_s \quad (2.35)$$

Then

$$\langle Z \rangle_u = \int_0^{f^{-1}(u)} f'(s) ds = u \quad (2.36)$$

which implies Z is a Brownian motion by Levy's characterization. Let $\tau = \in \{u \geq 0 : Z_u > K\}$ where $K > 0$ is a constant. Let $\pi_t = \sqrt{f'(t)}\mathbb{I}\big(t \leq f^{-1}(\tau)\big)$. Note that $\int_0^1 \pi_s^2 ds = \int_0^{f^{-1}(\tau)} f'(s)ds = \tau < \infty$. So $\int_0^t \pi_s dW_s$ makes sense for all $t \leq 1$. Let $X_t = \pi_s dW_s$, with $X_1 = \int_0^{f^{-1}(\tau)} \sqrt{f'(s)}dW_s = Z_\tau = K > 0$. X is a local martingale since it is a stochastic integral, but $\mathbb{E}(X_1) - K \neq X_0 = 0$.

Definition 2.28. An investment/consumption strategy (H, c) is L-admissible if $X_t(H, c) \geq -L_t$ for all t a.s. where L is given non-negative adapted process.

For most cases, $L = 0$.

Definition 2.29. A state price density is a positive Itô process such that $(Y_t P_t)_{t \geq 0}$ is a local martingale.

Theorem 2.30. *If there exists a state price density such that YL is uniformly integrable, then there is no arbitrage among L-admissible self-financing investment/consumption strategies.*

Remark 2.31. *Recall that $(Z_t)_{t\geq 0}$ is uniformly integrable if and only if*

$$\lim_{k\to\infty} \sup_{t\geq 0} \mathbb{E}(|Z_t|\mathbb{I}(Z_{t\geq k})) = 0 \tag{2.37}$$

Remark 2.32. *If $(Z_t)_{0\leq t\leq T}$ is a martingale then $(Z_t)_{0\leq t\leq T}$ is uniformly integrable ($T < \infty$ not random.)*

Remark 2.33. *If $\sup_{t\geq 0} \mathbb{E}(|Z_t|^p) < \infty$ for some $p > 1$ then $(Z_t)_{t\geq 0}$ is uniformly integrable.*

Remark 2.34. *If $Z_n \to Z_\infty$ a.s. and $(Z_n)_{n\geq 1}$ is UI then $\mathbb{E}(|Z_n - Z_\infty|) \to 0$.*

Proposition 2.35. *Let (H, c) be a self financing stragey and $X_t = H_t \cdot P_t$ so that $dX_t = H_t \cdot dP_t - c_t dt$. Let Y be an Itô process. Let Y be an Itô process. Then*

$$d(X_t Y_t) = H_t \cdot (dY_t P_t) - Y_t c_t dt. \tag{2.38}$$

Proof. Since $dX = H \cdot dP - cdt$, then

$$d\langle X, Y\rangle = \sum_{i=1}^n h^i d\left\langle P^i, Y^i \right\rangle \tag{2.39}$$

By Itô's formula,

$$d(XY) = XdY + YdX + d < X, Y > \tag{2.40}$$
$$= H \cdot PdY + Y(H \cdot dP - cdt) + \sum H^i d\left\langle P^i, Y^i \right\rangle \tag{2.41}$$
$$= \sum H^i (P^i dY + YdP^i + d\left\langle P^i, Y \right\rangle) - Ycdt \tag{2.42}$$
$$= \sum H^i d(P^i Y) - Ycdt \tag{2.43}$$

□

Definition 2.36. A continuous, adapted process $(Z_t)_{t\geq 0}$ is of class $\mathcal{D}$ (Doob) if $\{Z_\tau | \tau \text{ stopping times}\}$ is uniformly integrable.

Remark 2.37. *If* $\mathbb{E}\left(\sup_{t\geq 0} |Z_t|\right) < \infty$, *then* $(Z_t)_{t\geq 0}$ *is of class* $\mathcal{D}$.

Theorem 2.38. *If YL is of class D (at least locally), then there is no arbitrage.*

Theorem 2.39. *If there exists a state price density Y such that YL is of class D locally, then there are no L-admissible .*

Class D locally means $\{Z_{\tau\wedge t}$ - τ *a stopping time is* $UI \forall t \geq 0\}$.

Proof.

$$\int_0^t H_s \cdot d(X_s P_s) = Y_t X_t - Y_0 X_0 + \int_0^t Y_s c_s ds \tag{2.44}$$
$$\geq -Y_t L_t - Y_0 X_0 \tag{2.45}$$

if (H, c) is L-admissible. and from the lemma.

Also, since YP is a local martingale then $\int H \cdot d(YP)$ is a local martingale (by construction of the Itô integral), so there exists a sequence of stopping times $\tau_n \uparrow \infty$ such that $(\int H \cdot d(YP))^{\tau_n}$ is a true martingale.

Then

$$\mathbb{E}\left(\int_0^T H_s \cdot d(Y_s P_s) + Y_T L_T\right) = \mathbb{E}\left(\lim_{n\to\infty}\int_0^{T\wedge\tau_n} H_s \cdot d(Y_s P_s) + L_{T\wedge\tau_n} Y_{T\wedge\tau_n}\right) \tag{2.46}$$

$$\leq \liminf_{n\to\infty} \mathbb{E}\left(\int_0^{T\wedge\tau_n} Hd(YP) + L_{T\wedge\tau_n} Y_{T\wedge\tau_n}\right) \tag{2.47}$$

$$= \liminf_{n\to\infty} \mathbb{E}(Y_{T\wedge\tau_n} L_{T\wedge\tau_n}) \tag{2.48}$$

$$= \mathbb{E}(Y_T L_T) \tag{2.49}$$

by Fatau's lemma (2.47), using that $(\int_0^t H \cdot d(YP))^{\tau_n}$ is a martingale starting at zero (2.48) and the assumption of uniform integrability (2.49).

So suppose $X_0 = 0 = X_T$ almost surely. Then

$$\mathbb{E}\left(\int_0^T Y_s c_s ds\right) = \mathbb{E}\left(\int_0^T H_s \cdot d(Y_s P_s)\right) \leq 0 \Rightarrow c_t(\omega) = 0 a.e. \tag{2.50}$$

which implies no arbitrage. □

Suppose $P = (N, S)$ where $N_t > 0$ for all $t \geq 0$ almost surely - e.g. the price of a numeraire.

Definition 2.40. A pure investment strategy H is an arbitrage relative to the numeraire if and only if

(i) There exists a non-random $T > 0$ such that

$$\frac{X_T}{N_0} \geq \frac{N_T}{N_0} a.s. \tag{2.51}$$

and

$$\mathbb{P}\left(\frac{X_T}{N_0} > \frac{N_T}{N_0}\right) > 0 \tag{2.52}$$

Remark 2.41. *There exists a model P, credit limit L such that there is no absolute arbitrage but there is a relative arbitrage.*

To show

Definition 2.42. An equivalent (local) martingale measure is a measure $\mathbb{Q} \sim \mathbb{P}$ such that $\frac{S}{N}$ is a $\mathbb{Q}$-local martingale.

Theorem 2.43 (FTAP1 for market with a numeraire)**.** *Suppose $\mathbb{Q}$ is an EMM and $\frac{L}{N}$ is locally class D (with respect to $\mathbb{Q}$), then there are no L-admissible relative arbitrages.*

Lemma 2.44. *If $X_t = \phi_t N_t + \pi_t \cdot S_t$ (i.e (ψ, π) is a self-financing pure investment strategy), then*

$$d\frac{X_t}{N_t} = \pi_t d\frac{S_t}{N_t}. \tag{2.53}$$

Proof. Ito's lemma □

Proof (Proof of theorem). If $\mathbb{Q}$ is an EMM, X is a $\mathbb{Q}$-local martingale, since it is the stochastic integral with respect to the Q-local martingale $\frac{S}{N}$. As $\frac{X_t+L_t}{N_t} \geq 0$, we can apply Fatau's lemma as before, obtaining

$$\mathbb{E}_Q\left(\frac{X_T}{N_T}\right) \leq \frac{X_0}{N_0}. \tag{2.54}$$

Thus, if

$$\frac{X_T}{N_T} \geq \frac{X_0}{N_0} \tag{2.55}$$

$\mathbb{P}$ a.s. then

$$\frac{X_T}{N_T} \geq \frac{X_0}{N_0} \tag{2.56}$$

$\mathbb{Q}$ a.s by equivalence of $\mathbb{P}$ and $\mathbb{Q}$.

Then $\frac{X_T}{N_T} = \frac{X_0}{N_0}$ $\mathbb{Q}$ a.s. by the pigeon hole, then $\frac{X_T}{N_T} = \frac{X_0}{N_0}$ $\mathbb{P}$ a.s, since $\mathbb{P} \sim \mathbb{Q}$. □

Fill in rest of lecture content

In the framework $P = (B, S)$, $dB_t = B_t r_t dt$, $dS_t^i = S_i(\mu_t^i dt + \sum_{j=1}^m \sigma_t^{ij} dW_t^j)$.

Theorem 2.45. *Let λ_t be predictable and $\int_0^t \|\lambda_s\|^2 ds < 0$ a.s. $\forall t \geq 0$ and satisfying $\sigma_t \lambda_t = \mu_t = r_t$. Then $dY_t = -Y_t(r_t dt + \lambda_t dW_t)$ is a state price density and if W generates the filtration then all state price densities are of this form. λ is called a market price of risk.*

Proof. From Itô's formula,

$$d(Y_t B_t) = -Y_t B_t \lambda_t \cdot dW_t \tag{2.57}$$

is a local martingale,

$$d(Y_tS_t^i) = Y_tS_t^i(\mu_t^i + \sum \sigma^{ij} dW^j) + YS^i(-rdt - \sum \lambda^j dW^j) - YS^i \sum \sigma^{ij}\lambda^j dt \tag{2.58}$$

$$d(YS^i) = YS^i((\sigma^{ij} - \lambda)dW + (\mu^i - r - (\sigma\lambda)^i dt)) \tag{2.59}$$

Now, if the filtration is generated by W, then all positive local martingales M are of the form (by the martingale representation theorem) $dM = -M\lambda \cdot dW$ for some predictable process λ. So if Y is a state price density then Y is of the form $Y = \frac{M}{S}$ so $dY = -Y(rdt = \lambda dW)$. If YS^i is a local martingale for all i then $\sigma\lambda = u - r1$ in order for the dt to cancel in Itô's formula. □

If Y is a state price density such that YB is a true martingale, we can define an equivalent measure $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{Y_T B_T}{Y_0 B_0}$ for some fixed $T > 0$. This $\mathbb{Q}$ is an equivalent martingale measure.

Theorem 2.46. *Suppose $dM_t = -M_t\lambda_t \cdot dW_t$ is a true martingale where λ solves $\sigma\lambda = \mu - r1$. Fix $T > 0$ and let $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{M_T}{M_0}$. Then $\mathbb{Q}$ is an EMM and $dS_t^i = S_t^i(r_t dt + \sigma^{ij} d\hat{W}_t)$ for a $\mathbb{Q}$-Brownian motion $\hat{W}$.*

Proof. By Girsanov's theorem, $\hat{W}_t = W_t + \int_0^t \lambda_s ds$ is a $\mathbb{Q}$-Brownian motion. Now, by Itô,

$$d(\frac{S_i}{B}) = \frac{S_i}{B}((\mu^i - r)dt + \sigma^{ij} dW) \tag{2.60}$$

$$= \frac{S^i}{B}\sigma^{ij}(\lambda_t dt + dW_t) \tag{2.61}$$

$$= \frac{S^i}{B}\sigma^{ij} d\hat{W}. \tag{2.62}$$

□

Theorem 2.47. *Suppose that the filtration is generated by W. Suppose $n = d$ and that the $d \times d$ matrix $\sigma^{ij}(\omega)$ is invertible for all t, ω. Let $\lambda_t = \sigma_t^{ij}(\mu_t - r_t 1)$ and $dY_t = -Y_t(r_t dt + \lambda_t dW_t)$ is the unique state price density. Let ξ_T be a $\mathcal{F}_t$-measurable non-negative random variable such that $\xi_T Y_T$ is integrable. Then there exists a 0-admissible trading strategy H such that $X_T^H = \xi_T$ and $X_0^H = \frac{\mathbb{E}(Y_T\xi_T)}{Y_0}$.*

Furthermore, if LY is locally of class D and $\tilde{H}$ is an L-admissible strategy such that $X_T(\tilde{H}) = \xi_T$, then $X_0(\tilde{H}) \geq X_0(H)$. That is, $\frac{\mathbb{E}(Y_T\xi_T)}{\xi_0}$ is the *minimal replication cost of the European claim with payout ξ_T.*

Proof. Let $M_t = \mathbb{E}(Y_T \xi_T | \mathcal{F}_t)$. This is a martingale. We show that there exists H such that $X_t^H = \frac{M_t}{Y_t}$ for all $0 \leq t \leq T$. By the martingale representation theorem, there exists a d-dimensional predictable process α such that

$$dM_t = \alpha_t dW_t \tag{2.63}$$

By Itô's formula,

$$d\frac{M_t}{Y_t} = \frac{M_t}{Y_t} r_t dt + (\frac{M_t \lambda_t + \sigma_t}{Y_t})(dW_t + \lambda_t dt). \tag{2.64}$$

Let $\pi_t = \operatorname{diag}(S_t)^{-1}(\sigma_t^T)^{-1}(\frac{M_t\lambda_t+\sigma_t}{Y_t})$ and

$$\phi_t = \frac{\frac{M_t}{Y_t} - \pi_t S_t}{B_t}. \tag{2.65}$$

Note that $\phi_t B_t + \pi_t S_t = \frac{M_t}{Y_t}$, and

$$\pi_t dB_t + \pi_t dS_t = \frac{M_t}{Y_t} r dt + \frac{M_t \lambda_t + \alpha}{Y_t}(dW + \lambda dt) = d(\frac{M}{Y}) \tag{2.66}$$

and so $H = (\phi, \pi)$ is a self-financing strategy. It is 0-admissible since $\frac{M_t}{Y_t} > 0$. ☐

Theorem 2.48. *If $\tilde{H}$ is L-admissible and LY is in class D and $X_T(\tilde{H}) = \xi_T$ then*

$$X_0(\tilde{H}) \geq \frac{\mathbb{E}(Y_T \xi_T)}{Y_0} = X_0(H) \tag{2.67}$$

Proof. Consider

$$-Y_t(\tilde{X}_t + L_t) \geq 0 \tag{2.68}$$

and $Y_t \tilde{X}_t$ is a local martingale.

$$\mathbb{E}(Y_{T\wedge\tau_n} L_{T\wedge L_n}) \to \mathbb{E}(Y_T L_T) \tag{2.69}$$

by uniform integrability assumption. Therefore $Y\tilde{X}$ is a supermartingale by Fatau's lemma, and thus

$$\mathbb{E}(Y_T \xi_T) = \mathbb{E}\big(Y_T \tilde{X}_T\big) \leq Y_0 \tilde{X}_0 \tag{2.70}$$

□

Example 2.49. *A market model with no absolute arbitrage but with a relative arbitrage.*

Consider $P = (1, S)$, *where* $dS_t = S_t \sigma_t dW_t$, $n = d = 1$, $\sigma_t > 0$ *for all* t. *On the filtration generated by* W *and* S *is a strictly local martingale,* $\mathbb{E}(S_T) < S_0$ *(recall that all positive local martingales are supermartingales) which implies* $\mathbb{E}(\max_{0 \leq t \leq T} S_t) = \infty$.

Definition 2.50. Let $Y_t = 1$ for all t be a state price density. If L is of class D locally, there exist L-admissible absolute arbitrages.

Definition 2.51. Let $\mathbb{Q} = \mathbb{P}$. This is an EMM for the cash numeraire. If L is of class D locally, there are no relative arbitrages.

Definition 2.52. By existential replication theorem, there exists H such that $X_T(H) = S_T$. Notice that $X_0(H) = \mathbb{E}(X_T) < S_0$ (!)

Note that $\frac{X_T}{S_T} = 1$ a.s. but $\frac{X_0}{S_0} = p < 1$ (so we have a relative arbitrage). Let $\tilde{H} = H - p \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then

$$X_0(\tilde{H}) = \mathbb{E}(S_T) - pS_0 = 0 \tag{2.71}$$

$$X_T(\tilde{H}) = S_T - pS_T > 0 \tag{2.72}$$

$X_t(\tilde{H})$ is **not** of class D. So only admissible if L is wild.

3

Black-Scholes

Consider the market model

$$dB_t = B_t r dt \tag{3.1}$$

$$dS_t = S_t(\mu dt + \sigma dW_t) \tag{3.2}$$

Then $B_t = B_0 e^{rt}$, $S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$, and $Y_t = e^{-(r-\lambda^2 2)t - \lambda W_t}$ is the unique state price density with $Y_0 = 1$, where $\lambda = \frac{\mu - r}{\sigma}$.

Our goal is to replicate a European claim with payout $\xi_T = g(S_T)$ where $g \geq 0$ and suitably integrable. By our replication theorem, there exists a 0-admissible strategy H such that $X_t(H) = \frac{1}{Y_t}\mathbb{E}(Y_T g(S_T)|\mathcal{F}_t)$.

Let $\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\frac{\lambda^2 T}{2} - \lambda W_T}$ be the unique EMM. By the Cameron-Martin-Girsanov theorem, $\hat{W}_t = W_t + \lambda t$ is a $\mathbb{Q}$-Brownian motion. Then

$$S_T = S_t e^{(\mu - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)} \tag{3.3}$$

$$= S_t e^{(-r-\sigma^2 2)(T-t) + \sigma(\hat{W}_T - \hat{W}_t)} \tag{3.4}$$

and we have

$$X_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}(g(S_T)|\mathcal{F}_t) \tag{3.5}$$

$$= \int g(S_t e^{(r-\frac{\sigma^2}{2})(T-t) + \sigma\sqrt{T-t}Z})\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz \tag{3.6}$$

Substituting in $g(x) = (x-K)^+$ corresponding to a call option, we obtain the price

$$C_t(T,K) = S_t\Phi(\frac{-\log\frac{K}{S_t}}{\sigma\sqrt{T-t}} + (\frac{r}{\sigma} + \frac{\sigma}{2})\sqrt{T-t})$$
$$- Ke^{-r(T-t)}\Phi(\frac{-\log\frac{K}{S_t}}{\sigma\sqrt{T-t}} + (\frac{r}{\sigma} - \frac{\sigma}{2})\sqrt{T-t}) \quad (3.7)$$

Fill in missing lecture — Black-Scholes price as a solution to BS PDE

3.1 *Black-Scholes Volatility*

Assume we observe $(S_t)_{-T\le t\le 0}$ at some discrete intervals $(\frac{t}{n}-1)T$ for $i = 0, \dots, n$, with

$$Y_i = \log\frac{S_{t_i}}{S_{t_{i-1}}} \quad (3.8)$$
$$= (\mu - \frac{\sigma^2}{2})(t_i - t_{i-1}) + \sigma(W_{t_i} - W_{t_{i-1}}) \quad (3.9)$$
$$\sim N(a\frac{T}{n}, \frac{\sigma^2 T}{n}). \quad (3.10)$$

The MLE is then

$$\hat{a} = \frac{1}{T}\sum_{i=1}^{n} Y_i = \frac{1}{T}\log\frac{S_0}{S_{-T}} \quad (3.11)$$
$$\hat{\sigma}^2 = \frac{1}{T}\sum_{i=1}^{n}(Y_i - \frac{\hat{a}T}{n}) \quad (3.12)$$

and $\mathbb{V}(\hat{\sigma}^2) = \frac{2\sigma^4}{n} \to 0$ as $n \to \infty$.

3.2 *Calibration*

Black-Scholes model prediction, a call price

$$C_t(T,K) = C^{BS}(t,T,K,S_t,r,\sigma). \quad (3.13)$$

The Black-Scholes implied volatility for strike K, maturity T at time t is the unique σ which solves (3.13), denoted $\sum_t(T,K)$.

Black-Scholes predicts there is a unique number σ such that $\sum_t(T,K) = \sigma$ for all t, T, K. This fails in most markets.

3.3 Robustness

Consider a payout of claim $g(S_T)$. Assume we believe in Black-Scholes, and so we believe the price

$$V(0, S, \sigma) \tag{3.14}$$

where

$$V(t, S, \sigma) = e^{-r(T-t)} \int g(Se^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}z}) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz \tag{3.15}$$

for some σ. Pick $\hat{\sigma}$ to solve $V(0, S_0, \hat{\sigma}) = \xi_0$, the initial price of the claim.

Now, try to replicate the claim with portfolio (ϕ, π) with

$$\pi_t = \frac{\partial V}{\partial S}(t, S, \hat{\sigma}) \tag{3.16}$$

$$\phi_t = \frac{X_t - \pi_t S_t}{B_t} \tag{3.17}$$

Notice the equation

$$X_0 = V(0, S_0, \hat{\sigma}) \tag{3.18}$$

$$dX_t = r(X_t - \pi_t S_t)dt + \pi_t ds \tag{3.19}$$

has a unique solution given by

$$X_t = X_0 e^{rt} + e^{rt} \int_0^t \pi_s d(e^{-rs} S_s) \tag{3.20}$$

so given π, we can solve for X.

In the real model,

$$dB_t = rB_t dt \tag{3.21}$$

$$dS_t = S_t(\mu dt + \sigma_t dW_t) \tag{3.22}$$

for r, μ constant but σ_t a stochastic process.

Then

$$dV(t, S_t, \hat{\sigma}) = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} d\langle S \rangle \tag{3.23}$$

$$= (\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial^2 S} \sigma_t^2 S_t^2) dt + \pi_t dS_t \tag{3.24}$$

$$= (rV - rS \frac{\partial V}{\partial S} - \frac{1}{2} \frac{\partial^2 V}{\partial S^2} S^2 \hat{\sigma}^2 + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma_t^2 S_t^2) dt + \pi_t dS_t \tag{3.25}$$

and so

$$d(X_t - V(t, S_t, \hat{\sigma})) = r(X - V) dt + \frac{1}{2} S^2 (\hat{\sigma}^2 - \sigma_t^2) \frac{\partial^2 V}{\partial S^2} dt \tag{3.26}$$

and so

$$X_T - V(T, S_T, \hat{\sigma}) - X_0 + V(0, S_0, \hat{\sigma}) = X_T - g(S_T) \tag{3.27}$$

$$= \frac{1}{2} \int_0^T e^{-r(T-s)} S_s^2 (\hat{\sigma}^2 - \sigma_s^2) \frac{\partial^2 V}{\partial S^2} ds \tag{3.28}$$

and so we can estimate the difference between the option and the replicating portfolio by a weighted average of the gamma multiplied by the difference in implied and realized volatility over the time period.

4
Local Volatility Models

Consider

$$dB_t = rB_t dt \tag{4.1}$$

$$dS_t = S_t(\mu(t, S_t)dt + \sigma(t, S_t)dW_t) \tag{4.2}$$

$$= S_t(rdt + \sigma(t, S_t)d\hat{W}_t) \tag{4.3}$$

with $d\hat{W}_t = dW_t + \frac{\mu(t,S_t)-r}{\sigma(t,S_t)}dt$ is a Brownian motion under the equivalent martingale measure $\mathbb{Q}$.

Theorem 4.1 (Dupire). *Suppose $C_0(T, K) = \mathbb{E}_{\mathbb{Q}}\big(e^{-rT}(S_T - K)^+\big)$. Then*

$$\frac{\partial C_0}{\partial T} + rK\frac{\partial C_0}{\partial K} = \frac{\sigma(T, K)^2}{2}K^2\frac{\partial^2 C_0}{\partial K^2} \tag{4.4}$$

with $C_0(0, K) = (S_0 - K)^+$ with

$$\sigma(T, K) = \sqrt{\frac{2(\frac{\partial C_0}{\partial T} + rK\frac{\partial C_0}{\partial K})}{K^2\frac{\partial^2 C}{\partial K^2}}} \tag{4.5}$$

Exercise 4.2. *If*

$$C_0(T, K) = C^{BS}(t = 0, \sigma, T, S_0, K, r, \sigma_0) \tag{4.6}$$

show that

$$\sigma(T, K) = \sigma_0 \tag{4.7}$$

for all T, K.

Lemma 4.3 (Breden-Litzenberger, 1978). *Suppose S_T has density f (under $\mathbb{Q}$). Then*

$$C_0(T, K) = e^{-rT} \int_K^\infty f_{S_T}(y)(y-K)dy \tag{4.8}$$

$$\frac{\partial C_0}{\partial K} = -e^{-rT} \int_K^\infty f_{S_T}(y)dy \tag{4.9}$$

$$\frac{\partial^2 C_0}{\partial K^2} = e^{-rT} f_{S_T}(K) \tag{4.10}$$

Proof (Proof of Theorem 4.1). By Itô's formula,

$$(S_T - K^+) = (S_0 - K)^+ + \int_0^T \mathbb{I}(S_t \geq K)\, dS_t + \frac{1}{2}\int_0^T \delta_K d\langle S \rangle \tag{4.11}$$

$$= (S_0 - K)^+ + \int_0^T S_t r \mathbb{I}(S_t \geq K) + \frac{1}{2} S_t^2 \sigma(t, S_t)^2 \delta_K(S_t) dt + \int_0^T S_t \sigma(t, S_t) \mathbb{I}(S_t \geq K)\, d\hat{W}_t. \tag{4.12}$$

Taking $\mathbb{E}^{\mathbb{Q}}$ on both sides, we obtain

$$e^{rT} C_0(T, K) = (S_0 - K)^+ + \int_0^T \left(\int_K^\infty f_{S_t}(y) y r dy \right) dt + \frac{1}{2}\int_0^T f_{S_t}(K) K^2 \sigma(t, K)^2 dt \tag{4.13}$$

which gives

$$e^{rT} \frac{\partial C_0}{\partial T} + r e^{rT} C_0 = \int_K^\infty f_{S_T}(y) y r dy + \frac{1}{2} f_{S_T}(K) K^2 \sigma(T, K)^2 \tag{4.14}$$

Writing $y = (y - K) + K$ and applying the previous lemma, we obtain the required result. □

Remark 4.4. *Given a call surface $\{C_0(T, K), T, K > 0\}$ where $C_0(T, \cdot)$ is smooth, we find the density of S_T by*

$$\frac{\partial^2 C_0}{\partial K^2} = e^{-rT} f_{S_T}(K) \tag{4.15}$$

and hence

$$\mathbb{E}^{\mathbb{Q}}(e^{-rT} g(S_T)) = \int_0^\infty g(y) \frac{\partial^2 C_0}{\partial K^2}(T, y) dy \tag{4.16}$$

If g is convex and smooth, then

$$g(S_T) = g(a) + g'(a)(S-a) + \int_0^a g''(K)(K)(K-S_T)^+ dK + \int_a^\infty g''(K)(S_T-K)^+ dK \tag{4.17}$$

$$= \sum_{K_i \le a} g''(K_i)(K_i - S_T)^+ \Delta K_i + \sum_{K_i \ge a} g''(K_i)(S_T - K_i)\Delta K_i \tag{4.18}$$

4.1 Computing Moment Generating Functions

Consider a model with $B_t = B_0 e^{rT}$, S positive such that $(e^{-rT}S_t)_{t\ge 0}$ is a $\mathbb{Q}$-martingale.

Consider

$$\Theta = \{p + qi | 0 \le p \le i, q \in \mathbb{R}\} \subseteq \mathbb{C} \tag{4.19}$$

with $i = \sqrt{-1}$.

Let $M_t(\theta) = \mathbb{E}^{\mathbb{Q}} e^{\theta \log S_t}$ be the moment generating function of log S_t, with $\theta = p + iq$, $0 \le p \le 1$, and so

$$\mathbb{E}^{\mathbb{Q}} |e^{\theta \log S_t}| = \mathbb{E}^{\mathbb{Q}}(S_t^p) \le (\mathbb{E}^{\mathbb{Q}} S_t)^p = (e^{rt}S_0)^p < \infty \tag{4.20}$$

and so $M_t(\theta)$ is well defined for $\theta \in \Theta$.

Theorem 4.5.

$$\mathbb{E}^{\mathbb{Q}}(e^{-rT}(S_T-K)^+) = S_0 - \frac{e^{-rT}K^{1-p}}{2\pi} \int_{-\infty}^{\infty} \frac{M_T(p+ix)e^{-ix\log K}}{(x-ip)(x+i(1-p))} dx \tag{4.21}$$

for all $0 < p < 1$.

Theorem 4.6.

$$C_0(T,K) = S_0 \frac{e^{-rT}K^{1-p}}{2} \pi \int_{-\infty}^{\infty} \frac{M_T(p+ix)e^{-ix\log K}}{(x-ip)(x+i(1-p))} dx \tag{4.22}$$

Lemma 4.7.

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iax}}{x-ip} x + i(1-p) = \begin{cases} e^{-ap} & a \ge 0 \\ a^{a(1-p)} & a < 0 \end{cases} \tag{4.23}$$

which can be shown via contour integration.

Let γ_R be the semi-circle of radius R above the x-axis in the complex plane. Then

$$\int_{\gamma_R} \frac{e^{iax}}{(x-ip)(x+i(1-p))} dx = 2\pi \operatorname*{Res}_{x=ip} = 2\pi e^{-ap}. \tag{4.24}$$

and we have

$$\int_{-R}^{R} + \int_{\phi=0}^{\pi} \frac{e^{ia(R\cos\phi + i\sin\phi)}}{(Re^{i\phi}-ip)(Re^{i\phi}+i(1-p))} d\phi \leq \frac{e^{-aR\sin\phi}}{\frac{1}{2}R} \to 0 \tag{4.25}$$

and so we obtain our required result.

Proof (Proof of 4.6). We have

$$\begin{aligned} e^{-rT}(S_T-K)^+ &= e^{-rT}S_T \\ &\quad - \frac{K^{1-p}e^{-rT}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{p\log S_T + ix\log S_T - ix\log K}}{(x-ip)(x+i(1-p))} dx \end{aligned} \tag{4.26}$$

Now computing $\mathbb{E}^{\mathbb{Q}}$, using Fubini's theorem to justify the interchange as

$$\mathbb{E}\left(\int \left| \frac{e^{(p+ix)\log S_T - ix\log K}}{(x-ip)(x+i(1-p))} \right| dx\right) = M_T(p) \int \frac{1}{\sqrt{(x^2+p^2)(x^2+(1-p)^2)}} < \infty \tag{4.27}$$

$\square$

Remark 4.8. *By Holder's inequality, $p \mapsto \log M_T(p) = \Lambda_T(p)$ is convex. $\Lambda_T(0) = 0, \Lambda_T(1) = \log S_0 + rT$, and $p \mapsto \Lambda_T(p)$ is smooth. It has a minimal point $p = p^\star \in (0,1)$ at*

$$\Lambda_T(p^\star + ix) \approx \Lambda_T(p^\star) + \Lambda_T'(p^\star)(ix) + \frac{1}{2} \underbrace{\Lambda''}_{\geq 0 \text{ by convexity}} (p^\star)(ix)^2 \tag{4.28}$$

$$= \ldots \tag{4.29}$$

by Taylor's theorem.

Then

$$\int \frac{M_T(p^\star + ix)e^{-ix\log K}}{(x-ip)(x+i(1-p))} \approx M_T(p^\star)\int \frac{e^{-\Lambda_T''(p^\star)x^2}}{p(1-p)}dx \tag{4.30}$$

$$= \frac{M_T(p^\star)}{p(1-p)}\sqrt{\frac{2\pi}{\Lambda_T''(p^\star)}} \tag{4.31}$$

4.2 *The Heston Model*

$$dB_t = B_t r dt \tag{4.32}$$

$$dS_t = S_t(rdt + \sqrt{v_t dW_t^S}) \tag{4.33}$$

$$dv_t = \lambda(\overline{v} - v_t)dt + c\sqrt{v_t}dW_t^V \tag{4.34}$$

W^S, W^v are Brownian motions under some EMM $\mathbb{Q}$, with correlation ρ. For instance, $W_t^v = \rho W_t^s + \sqrt{1-\rho^2}d_t^\perp$ with $W^s, W^\perp$ independent.

$\overline{v} > 0$ is the mean-reversion level. $\lambda > 0$ is the mean reversion rate. We have $v_t \geq 0$ almost surely [Cox et al., 1985].

Our goal is fix $T > 0, \theta \in \Theta$, want to compute $\mathbb{E}\left(e^{\theta \log S_T}\right)$.

Idea: Let $(V(t, S_t, v_t))_{0 \leq t \leq T}$ be chosen so that it is a martingale with $V(T, S_T, V_T) = e^{\theta \log S_T}$. The moment generating function is then $V(t = 0, S_0, v_0)$.

By Itô,

$$dV(t, S_{t,v_t}) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}d\langle S\rangle + \frac{\partial V}{\partial v}dv + \frac{1}{2}\frac{\partial^2}{\partial v^2}d\langle v\rangle + \frac{\partial^2 V}{\partial v \partial s}d\langle S, v\rangle. \tag{4.35}$$

We seek to make the dt terms vanish. Thus,

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}rS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2 v + \frac{\partial V}{\partial v}\lambda(\overline{v} - v) + \frac{1}{2}\frac{\partial^2 V}{\partial v^2}c^2 v + \frac{\partial^2 V}{\partial S \partial v}\rho S v c = 0. \tag{4.36}$$

The inspired idea is to look for solutions of the form

$$V(t, S, v) = e^{\theta \log S + R(T-t)v + Q(T-t)} \tag{4.37}$$

with $R(0) = Q(0) = 0$.

Substituting this functional form in, we obtain

$$R'v - Q' + r\theta + \frac{1}{2}\theta(\theta-1)v + R\lambda(\overline{v}-v) + \frac{1}{2}R^2c^2v + \theta R\rho vc = 0 \tag{4.38}$$

Collecting terms, we have

$$\begin{cases} R' = \frac{1}{2}\theta(\theta-1) + \frac{1}{2}R^2c^2 + (\theta pc - \lambda)R \\ Q' = r\theta = R\lambda\overline{v} \end{cases} \tag{4.39}$$

which are Riccati equations, which have an explicit solution.

4.3 American Options (Guest Lecture)

Suppose we have some assets d and our bank account B_t. The random assets evolve as

$$dS_t^i S_t^i(\mu_t^i dt + \sum_{j=1}^d \sigma_{ij}(t, S_t)dW_t^j) \tag{4.40}$$

The option we want to price pays $g(S_\tau)$ if exercised at time τ. The exercise time τ must be a stopping time, with $\tau \leq T$, the expiration time.

For technical reasons, suppose g is bounded. For examples sake, we assume we have one sock, and consider an American put $g(S) = (K-S)^+$.

If there are d assets, we might have a min-put, we have

$$g(S) = (K - \min_{1\leq i\leq d} S^i)^+ = \max_{1\leq i\leq d}(K-S^i)^+ \tag{4.41}$$

To solve this pricing problem, write

$$\mathcal{L}f = \frac{1}{2}\sum_{i,j} S_iS_ja_{ij}(t,S)\frac{\partial^2 f}{\partial S_i \partial S_j} + \sum_i rS_i\frac{\partial f}{\partial S_i} - rf + \frac{\partial f}{\partial t} \tag{4.42}$$

where $a = \sigma\sigma^T$, and suppose we can find some $V(t,S) \in C^{1,2}$ such that

$$\max\{\mathcal{L}V, g - V\} = 0, V(T, \cdot) = g(\cdot). \tag{4.43}$$

Then

$$V(0, S_0) = \sup_{\tau \leq T} \mathbb{E}\big(e^{-r\tau} g(S_\tau) | S_0\big) \tag{4.44}$$

Why is this true? Consider

$$d(V(t, S_t)e^{-rt}) = V_s(t, S_t) S_t \sigma_t dW_t + \mathcal{L}V(t, S_t)dt \tag{4.45}$$

If we let τ be any stopping time $\leq T$, and we let $T \uparrow \infty$ be a sequence of stopping times "rediscovering" the local martingale $V_S(t, S)S\sigma dW$, and we shall then have

$$V(0, S_0) = \mathbb{E}\left(e^{-r\tau_n} V(\tau_n, S_{\tau_n}) - \int_0^{\tau_n} \mathcal{L}V(u, S_u)du\right) \tag{4.46}$$

$$\geq \mathbb{E}\big(e^{-r\tau_n} V(\tau_n, S_{\tau_n})\big) \tag{4.47}$$

$$\geq \mathbb{E}\big(e^{-r\tau_n} g(S_{\tau_n})\big). \tag{4.48}$$

since $\mathcal{L}V \leq 0$.

If we let $n \to \infty, \tau_n \uparrow \tau$, we must have that

$$V(0, S_0) \geq \sup_{0 \leq \tau \leq T} \mathbb{E}\big(e^{-r\tau} g(S_\tau)\big). \tag{4.49}$$

To show that there is equality, consider

$$\tau^\star = \inf\{t | V(t, S_t) = g(S_t)\} \tag{4.50}$$

We know that $V(T, \cdot) = g(\cdot)$, and so $\tau^\star \leq T$. We also notice that in $[0, \tau)$, $\mathcal{L}V = 0$ because in $[0, \tau)$, $g - V < 0$, and $\max\{\mathcal{L}V, g - V\} = 0$. Now going back to the first calculation, if we write $\tau_n^\star = \tau^\star \wedge T_n$.

$$
\begin{aligned}
V(0,S_0) &= \mathbb{E}\left(e^{-r\tau_n^\star}V(\tau_n^\star, S_{\tau_n^\star}) - \int_0^{\tau_n^\star}\mathcal{L}V(u,S_u)du\right) && (4.51)\\
&= \mathbb{E}\left(e^{-r\tau_n}V(\tau_n, S_{\tau_n})\right) && (4.52)\\
&= \mathbb{E}\left(e^{-r\tau^\star}V(\tau^\star, S_{\tau^\star}) : \tau^\star \le T_n\right) + \mathbb{E}\left(e^{-rT_n}V(T_n, S_{T_n}) : \tau^\star > T_n\right) && (4.53)\\
&= \mathbb{E}\left(e^{-r\tau^\star}g(S_{\tau^\star})|\tau^\star \le T_n\right) + \mathbb{E}\left(e^{-rT_n}V(T_n, S_{T_n}) : \tau^\star > T_n\right) && (4.54)\\
&\to \mathbb{E}\left(e^{-r\tau^\star}g(S_{\tau^\star})\right). && (4.55)
\end{aligned}
$$

n We need to show that the V we found is bounded.

Example 4.9. *American puts in one dimension.*

We have an envelope V.

We find V by solving

$$0 = -rV = \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_s \tag{4.56}$$

for $S = q$ with boundary condition

$$V(q) = (K-q)^+ \tag{4.57}$$

This we can write as

$$V(S) = AS + BS^{-2r/\sigma^2} \tag{4.58}$$

with the boundary condition $V(q) = (K-q)^+$.

Suppose we let q be a parameter of the stopping rule, work out the value and optimize over q. The value is

$$V(S) = (K-q)(\frac{S}{q})^{-\frac{2r}{\sigma^2}} = S^{-\frac{2r}{\sigma^2}}q^{\frac{2r}{\sigma^2}}(K-q) \tag{4.59}$$

Optimizing over q, we have

$$\frac{2r}{\sigma^2 q} = \frac{1}{K-q} \Rightarrow q = \frac{2rk}{\sigma^2+2r}. \tag{4.60}$$

We can check, if we use this value of q, then $V'(q) = -1 = \frac{\partial}{\partial S}(K - S)|_{s=q}$.

It can be shown that $\sup_{0\leq\tau\leq T}\mathbb{E}(e^{-r\tau}g(S_\tau))\leq\min_{M\in\mathcal{M}_0}\mathbb{E}(\sup_{\dots})$ Fill in from lecture notes.?

5

Bond Markets and Interest Rates

Definition 5.1. A zero coupon bond is a contingent claim that pays exactly one unit of money at maturity.

We assume that ξ_T, the payment of the bond, is 1 a.s. - that is, there is no credit risk.

Definition 5.2. $P(t, T)$is the price at time t for a bond maturing at time T.

Definition 5.3. The yield $y(t, T)$ is defined as

$$y(t,T) = -\frac{1}{T-t}\log P(t,T) \tag{5.1}$$

or equivalently

$$P(t,T) = e^{-(T-t)y(t,T)} \tag{5.2}$$

Definition 5.4. We call $\lim_{T\downarrow t} y(t, T) = r_t$ the "spot" or "short" rate.

We call $\lim_{T\uparrow\infty} y(t, T)$ if it exists.

Definition 5.5. The forward rate $f(t, T)$ is defined

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T) \tag{5.3}$$

or equivalently

$$P(t,T) = -\int_t^T f(t,u)du \tag{5.4}$$

Theorem 5.6. *There is no arbitrage in the market prices* $(P(t, T_1), P(t, T_2), \ldots, P(t, T_n))$ *if* $Y_t P(t, T)_{t\in[0,T]}$ *is a local martingale for all* T*, where* Y *is a state price* *density.*[1]

In particular, there is no arbitrage if $P(t,T) = \frac{1}{Y_t}\mathbb{E}(Y_T|\mathcal{F}_t)$

Introduce the bank account $dB_t = B_t r_t dt \iff B_t = B_0 e^{\int_0^t r_s ds}$ where r is the short rate. Define an equivalent martingale measure with density $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{B_T Y_T}{B_0 Y_0}$. Rewrite

$$P(t,T) = B_t \mathbb{E}_{\mathbb{Q}}\left(\frac{1}{B_T}|\mathcal{F}_t\right) = \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}|\mathcal{F}_t\right) \tag{5.5}$$

By the law of one price,

$$f(t,T) = -\frac{\partial}{\partial T}\log \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}|\mathcal{F}_t\right) \tag{5.6}$$

$$= \frac{\mathbb{E}_{\mathbb{Q}}\left(r_T e^{-\int_t^T r_s ds}|\mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}|\Phi_t\right)}, \tag{5.7}$$

and so $f(t,T)$ can be seen as the "market weighted conditional expectation of r_T given at $\mathcal{F}_t$."

Alternatively, we have

$$\mathbb{E}_{\mathbb{Q}}\left((f(t,T) - r_T)e^{-\int_t^T r_s ds}|\mathcal{F}_t\right) = 0 \tag{5.8}$$

and so the forward rate is such that the claim with payout $f(t,T) - r_T$ has price 0 at time T.

Fill in missing lecture from Monday 2 December

There are two approaches to bond market pricing:

(i) Let $(r_t)_{t\geq 0}$ be fundamental, derive everything else: $f(t,T)$, etc.

(ii) Model $(f(t,T))_{0\leq t\leq T}$ directly - the Heath et al. [1992] approach.

5.1 *The Heath et al. [1992] Model*

Theorem 5.7. *Suppose* $df(t,T) = a(t,T)dt + \sigma(t,T)\cdot d\hat{W}_t$ *for a d-dimensional Brownian motion* $\hat{W}$ *where* $\sigma(t,T)$ *is suitably measurable and integrable, and*

$$a(t,T) = \sigma(t,T)\cdot\int_t^T \sigma(t,u)du \tag{5.9}$$

[1] Recall relative arbitrage, admissible class D, etc.

Define $r_t = f(t,t)$ *and* $P(t,T) = e^{-\int_t^T f(t,u)du}$. *Then*

$$\left(e^{-\int_0^t r_s ds} P(t,T)\right)_{0 \le t \le T} \tag{5.10}$$

is a local martingale.

Remark 5.8.

$$f(t,T) = f(0,T) + \int_0^t a(s,T)ds + \int_0^t \sigma(s,T) \cdot d\hat{W}_s. \tag{5.11}$$

Proof. Recall that if $d \log M_t = -\frac{|b_t|^2}{2}dt + b_t \cdot d\hat{W}_t$, then M is a local martingale if and only if $M_t = M_0 e^{-\frac{1}{[}2] \int_0^t |b_s|^2 ds + \int_0^t b_s \cdot d\hat{W}_s}$.

By differentiation, we have

$$d\left(-\int_0^t r_s ds - \int_t^T f(t,u)du\right) = -r_t dt + f(t,t)dt - \int_t^T df(t,u)du \tag{5.12}$$

$$= -(\int_t^T a(t,u)du)dt - (\int_t^T \sigma(t,u)du) \cdot d\hat{W}_t. \tag{5.13}$$

noting that

$$\int_t^T a(t,u)du = \frac{1}{2} \| \int_t^T \sigma(t,u)du \|^2 \tag{5.14}$$

gives the required result. □

Example 5.9 (Ho and Lee [1986]). *Assume* $d = 1$, $\sigma(t,T) = \sigma_0$ *constant. Then*

$$df(t,T) = ((T-t)\sigma_0^2)dt + \sigma_0 d\hat{W}_t \tag{5.15}$$

$$f(t,T) = f(0,T) + \int_0^t (T-s)\sigma_0^2 ds + \sigma_0 d\hat{W}_t \tag{5.16}$$

$$r_t = f(0,t) + \frac{1}{2}\sigma_0^2 t^2 + \sigma_0 \hat{W}_t \tag{5.17}$$

Example 5.10 (Hull and White [1990]). *Again, assume* $d = 1$, $\sigma(t,T) = \sigma_0 e^{-\lambda(T-t)}$.

$$df(t,T) = \sigma_0^2 e^{-\lambda(T-t)}(1 - e^{-\lambda(T-t)})dt + \sigma_0 e^{-\lambda(T-t)} d\hat{W}_t \tag{5.18}$$

$$dr_t = \lambda\left(\frac{f_0'(t)}{\lambda} + f_0(t) + \frac{\sigma_0^2}{2\lambda^2}(1 - e^{-\lambda t}) - r_t\right) + \sigma_0 d\hat{W}_t. \tag{5.19}$$

Example 5.11 (Kennedy [1997]). *This is a Gaussian random field model. Suppose $\sigma(t, T)$ is not random, so*

$$f(t, T) = f(0, T) + \int_0^T a(s, T) ds + \int_0^t \sigma(s, T) d\hat{W}_s \tag{5.20}$$

is Gaussian. Then

$$\mathbb{E}_{\mathbb{Q}}(f(t, T)) = f(0, T) + \int_0^t a(s, T) ds \tag{5.21}$$

$$Cov(f(s, S), f(t, T)) = \int_0^{s \wedge t} \sigma(u, S) \cdot \sigma(u, T) du \tag{5.22}$$

Turning this around, we can model

$$(f(t, T))_{0 \leq t \leq T} \tag{5.23}$$

as a Gaussian random field with

$$Cov(f(s, S), f(t, T)) = c_{s \wedge t}(S, T) \tag{5.24}$$

$$\mathbb{E}(f(t, T)) = f(0, T) + \int_0^T c_{s \wedge t}(s, T) ds, \tag{5.25}$$

and thus there is no need to introduce a Brownian motion. For instance,

$$d\langle f(t, S), f(t, T) \rangle = \sigma(t, S) \cdot \sigma(t, T) dt \tag{5.26}$$

$$= \sigma_0 e^{-\beta |T - S|} \tag{5.27}$$

and so we have an exponentially decaying correlation between forward rates of different maturities.

Example 5.12. *The HJM equation*

$$df(t, T) = a(t, T) dt + \sigma(t, T) dW_t \tag{5.28}$$

$$T = t + x, f_t(x) = f(t, t + x) \tag{5.29}$$

$$df_t(x) = (\frac{\partial f}{\partial x} + a_t(x)) dt + \sigma_t(x) dW_t \tag{5.30}$$

Fix a separable Hilbert space $F = \{f : \mathbb{R}_+ \to \mathbb{R}\}$. Then (dropping the x),

$$df_t = (Af_t + \alpha_t)\, dt + \sigma_t dW_t \tag{5.31}$$

can be interpreted as an evolution equation in this function space. In the simplest case, σ_t *is a constant vector* $F \otimes \mathbb{R}^d$*,* α_t *is a constant vector in* F*, then* $(f_t)_{t \geq 0}$ *is an F-valued Ornstein-Uhlenbeck process.*

We can apply techniques from statistics (e.g. PCA) if this model has an invariant measure — shown in early 2000's.

6

Bibliography

John C Cox, Jonathan E Ingersoll Jr, and Stephen A Ross. A theory of the term structure of interest rates. *Econometrica: Journal of the Econometric Society*, pages 385–407, 1985.

David Heath, Robert Jarrow, and Andrew Morton. Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica: Journal of the Econometric Society*, pages 77–105, 1992.

Thomas SY Ho and Sang-Bin Lee. Term structure movements and pricing interest rate contingent claims. *The Journal of Finance*, 41(5): 1011–1029, 1986.

John Hull and Alan White. Pricing interest-rate-derivative securities. *Review of financial studies*, 3(4):573–592, 1990.

Douglas P Kennedy. Characterizing gaussian models of the term structure of interest rates. *Mathematical Finance*, 7(2):107–118, 1997.