

MSH7 - APPLIED PROBABILITY AND STOCHASTIC CALCULUS

ANDREW TULLOCH

CONTENTS

1. Lecture 1 - Tuesday 1 March	2
2. Lecture 2 - Thursday 3 March	2
2.1. Concepts of convergence	2
3. Lecture 3 - Tuesday 8 March	3
4. Lecture 4 - Thursday 10 March	5
5. Lecture 5 - Tuesday 15 March	6
5.1. Properties of paths of Brownian motion	7
6. Lecture 6 - Thursday 17 March	9
7. Lecture 7 - Tuesday 22 March	10
7.1. Construction of Brownian motion	10
8. Lecture 8 - Thursday 24 March	11
9. Lecture 9 - Tuesday 29 March	12
10. Lecture 10 - Thursday 31 March	15
11. Lecture 11 - Tuesday 5 March	16
12. Lecture 12 - Thursday 7 March	19
13. Lecture 13 - Tuesday 12 March	20
14. Lecture 14 - Thursday 14 March	22
15. Lecture 15 - Tuesday 19 March	24
16. Lecture 16 - Thursday 21 March	26
17. Lecture 17 - Tuesday 3 May	26
17.1. Stochastic integrals for martingales	26
18. Lecture 18 - Tuesday 10 May	30
18.1. Itô's integration for continuous semimartingales	31
18.2. Stochastic differential equations	32
19. Lecture 19 - Thursday 12 May	32
20. Lecture 20 - Tuesday 17 May	33
20.1. Numerical methods for stochastic differential equations	33
20.2. Applications to mathematical finance	34

20.3. Martingale method	34
21. Lecture 21 - Thursday 19 May	35
21.1. Change of Measure	35
22. Lecture 22 - Tuesday 24 May	36
22.1. Black-Scholes model	38
23. Lecture 23 - Thursday 26 May	38
24. Lecture 24 - Tuesday 31 May	40
25. Lecture 25 - Thursday 2 June	42

1. LECTURE 1 - TUESDAY 1 MARCH

Definition 1.1 (Finite dimensional distribution). The **finite dimensional distribution** of a stochastic process X is the joint distribution of $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$

Definition 1.2 (Equality in distribution). Two random variables X and Y are **equal in distribution** if $\mathbb{P}(X \leq \alpha) = \mathbb{P}(Y \leq \alpha)$ for all $\alpha \in \mathbb{R}$. We write $X \stackrel{d}{=} Y$.

Definition 1.3 (Strictly stationary). A stochastic process X is **strictly stationary** if

$$(X_{t_1}, X_{t_2}, \dots, X_{t_n}) = (X_{t_1+h}, X_{t_2+h}, \dots, X_{t_n+h})$$

for all t_i, h .

Definition 1.4 (Weakly stationary). A stochastic process X is **weakly stationary** if $\mathbb{E}(X_t) = \mathbb{E}(X_{t+h})$ and $\text{Cov}(X_t, X_s) = \text{Cov}(X_{t+h}, X_{s+h})$ for all t, s, h .

Lemma 1.5. If $\mathbb{E}(X_t^2) < \infty$, then strictly stationary implies weakly stationary.

Example 1.6.

- The stochastic process $\{X_t\}$ with X_t all IID is strictly stationary.
- The stochastic process W_t with $W_t \sim N(0, t)$ and $X_t - X_s$ independent of X_s (for $s < t$) is not strictly or weakly stationary.

Definition 1.7 (Stationary increments). A stochastic process has **stationary increments** if

$$X_t - X_s \stackrel{d}{=} X_{t-h} - X_{s-h}$$

for all s, t, h .

2. LECTURE 2 - THURSDAY 3 MARCH

Example 2.1. Let $X_n, n \geq 1$ be IID random variables. Consider the stochastic process $\{S_n\}$ where $S_n = \sum_{j=1}^n X_j$. Then $\{S_n\}$ has stationary increments.

2.1. **Concepts of convergence.** There are three major concepts of convergence of random variables.

- Convergence in distribution
- Convergence in probability
- Almost surely convergence

Definition 2.2 (Convergence in distribution). $X_n \xrightarrow{d} X$ if $\mathbb{P}(X_n \leq x) \rightarrow \mathbb{P}(X \leq x)$ for all x .

Definition 2.3 (Convergence in probability). $X_n \xrightarrow{P} X$ if $\mathbb{P}(|X_n - X| \geq \epsilon) \rightarrow 0$ for all $\epsilon > 0$.

Definition 2.4 (Almost surely convergence). $X_n \xrightarrow{a.s.} X$ if except on a null set A , $X_n \rightarrow X$, that is, $\lim_{n \rightarrow \infty} X_n = X$. And hence $\mathbb{P}(\lim_{n \rightarrow \infty} X_n = X) = 1$

Definition 2.5 (σ -field generated by X). Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We call $\sigma(X)$ the σ -field generated by X , and we have

$$\sigma(X) = \{X^{-1}(B) : B \in \mathcal{B}\}$$

where $X^{-1}(B) = \{\omega, X(\omega) \in B\}$ and \mathcal{B} is the Borel set on \mathbb{R} .

Definition 2.6 (Conditional probability). We have $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ if $\mathbb{P}(B) \neq 0$.

Definition 2.7 (Naive conditional expectation). We have $\mathbb{E}(X|B) = \frac{\mathbb{E}(X \mathbb{1}_B)}{\mathbb{P}(B)}$.

Definition 2.8 (Conditional density). Let $g(x, y)$ be the joint density function for X and Y . Then we have $\int_{\mathbb{R}} g(x, y) dx \equiv g_Y(y)$. We also have

$$g_{X|Y=y} = \frac{g(x, y)}{g_Y(y)}$$

which defines the conditional density given $Y = y$.

Finally, we define $\mathbb{E}(X|Y = y) = \int_{\mathbb{R}} x g_{X|Y=y}(x) dx$.

Definition 2.9 (Conditional expectation). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let \mathcal{A} be a sub σ -field of \mathcal{F} . Let X be a random variable such that $\mathbb{E}(|X|) < \infty$. We define $\mathbb{E}(X|\mathcal{A})$ to be a random variable Z such that

- (i) Z is \mathcal{A} -measurable,
- (ii) $\mathbb{E}(X \mathbb{1}_A) = \mathbb{E}(Z \mathbb{1}_A)$ for all $A \in \mathcal{A}$.

Proposition 2.10 (Properties of the conditional expectation). Consider $Z = \mathbb{E}(X|Y) = \mathbb{E}(X|\sigma(Y))$

- If T is $\sigma(Y)$ -measurable, then $\mathbb{E}(XT|Y) = T\mathbb{E}(X|Y)$ a.s.
- If T is independent of Y , then $\mathbb{E}(T|Y) = \mathbb{E}(T)$.
- $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|T))$

3. LECTURE 3 - TUESDAY 8 MARCH

Definition 3.1 (Martingale). Let $\{X_t, t \geq 0\}$ be a **right-continuous** with **left-hand limits**.

$$\lim_{t \uparrow t_0} X_t \text{ exists}$$

Let $\{\mathcal{F}_t, t \geq 0\}$ be a filtration.

Then X is called a martingale with respect to \mathcal{F}_t if

- (i) X is **adapted to** \mathcal{F}_t , i.e. X_t is \mathcal{F}_t -measurable
- (ii) $\mathbb{E}(|X|) < \infty, t \geq 0$
- (iii) $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ a.s.

Example 3.2. Let X_n be IID with $\mathbb{E}(X_n) = 0$. Then $\{S_k, k \geq 0\}$, where $S_k = \sum_{i=0}^k X_i$, is a martingale.

Example 3.3. An independent increment process $\{X_t, t \geq 0\}$ with $\mathbb{E}(X_t) = 0$ and $\mathbb{E}(|X_t|) \leq \infty$ is a martingale with respect to $\mathcal{F}_t = \sigma\{X_s, 0 \leq s \leq t\}$

Definition 3.4 (Gaussian process). Let $\{X_t, t \geq 0\}$ be a stochastic process. If the finite dimensional distributions are multivariate normal, that is,

$$(X_{t_1}, \dots, X_{t_m}) \equiv N(\mu, \Sigma)$$

for all t_1, \dots, t_m , then we call X_t a **Gaussian process**

Definition 3.5 (Markov process). A continuous time process X is a **Markov process** if for all t , each $A \in \sigma(X_s, s > t)$ and $B \in \sigma(X_s, s < t)$, we have

$$\mathbb{P}(A|X_t, B) = \mathbb{P}(A|X_t)$$

Definition 3.6 (Diffusion process). Consider the stochastic differential equation

$$dX_t = \mu(t, x)dt + \sigma(t, x)dB_t$$

A diffusion process is **path-continuous, strong Markov** process such that

$$\begin{aligned} \lim_{h \rightarrow 0} h^{-1} \mathbb{E}(X_{t+h} - X_t | X_t = x) &= \mu(t, x) \\ \lim_{h \rightarrow 0} h^{-1} \mathbb{E}([X_{t+h} - X_t - h\mu(t, X)]^2 | X_t = x) &= \sigma^2(t, x) \end{aligned}$$

Definition 3.7 (Path-continuous). A process is **path-continuous** if $\lim_{t \rightarrow t_0} X_t = X_{t_0}$.

Definition 3.8 (Lévy process). Let $\{X_t, t \geq 0\}$ be a stochastic process. We call X a **Lévy process**

- (i) $X_0 = 0$ a.s.
- (ii) It has stationary and independent increments
- (iii) X is stochastically continuous, that is, for all $s, t, \epsilon > 0$, we have

$$\lim_{s \rightarrow t} \mathbb{P}(|X_s - X_t| \geq \epsilon) = 0.$$

Equivalently, $X_s \xrightarrow{P} X_t$ if $s \rightarrow t$.

Example 3.9 (Poisson process). Let $(N(t), t \geq 0)$ be a stochastic process. We call $N(t)$ a **Poisson process** if the following all hold:

- (i) $N(0) = 0$
- (ii) N has independent increments
- (iii) For all $s, t \geq 0$,

$$\mathbb{P}(N(t+s) - N(s) = n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!} \quad n = 0, 1, 2, \dots$$

The Poisson process is stochastically continuous - that is,

$$\begin{aligned}\mathbb{P}(|N(t) - N(s)| \geq \epsilon) &= \mathbb{P}(|N(t-s) - N(0)| \geq \epsilon) \\ &= 1 - \mathbb{P}(|N(t-s)| < \alpha) \\ &= 1 - \mathbb{P}(|N(t-s)| = 0) \\ &= 1 - e^{-\lambda(t-s)} \rightarrow 0 \quad \text{as } t \rightarrow s\end{aligned}$$

The Poisson process is **not** path-continuous, that is

$$\mathbb{P}(\lim_{t \rightarrow s} |N(t) - N(s)| = 0) \neq 1$$

because

$$\mathbb{P}(\cup_{|t-s| \geq \epsilon} |N(t) - N(s)| > \delta) \geq \mathbb{P}(|N(s+1) - N(s)| \geq \delta) > 0$$

4. LECTURE 4 - THURSDAY 10 MARCH

Definition 4.1 (Self-similar process). For any $t_1, t_2, \dots, t_n \geq 0$, for any $c > 0$, there exists an H such that

$$(X_{ct_1}, X_{ct_2}, \dots, X_{ct_n}) \stackrel{d}{=} (c^H X_{t_1}, c^H X_{t_2}, \dots, c^H X_{t_n}).$$

We call H the **Hurst index**.

Example 4.2 (Fractional process).

$$\begin{aligned}(1-B)^d X_t &= \epsilon_t, \quad \epsilon_t \text{ martingale difference} \\ BX_t &= X_{t-1}, \quad 0 < d < 1\end{aligned}$$

Definition 4.3 (Brownian motion). Let $\{B_t, t \geq 0\}$ be a stochastic process. We call B_t a **Brownian motion** if the following hold:

- (i) $B_0 = 0$ a.s.
- (ii) $\{B_t\}$ has stationary, independent increments.
- (iii) For any $t > 0$, $B_t \equiv N(0, t)$
- (iv) The path $t \mapsto B_t$ is continuous almost surely, i.e.

$$\mathbb{P}(\lim_{t \rightarrow t_0} B_t = B_{t_0}) = 1$$

Definition 4.4 (Alternative formulations of Brownian motion). A process $\{B_t, t \geq 0\}$ is a Brownian motion if and only if

- $\{B_t, t \geq 0\}$ is a Gaussian process
- $\mathbb{E}(B_t) = 0, \mathbb{E}(B_s B_t) = \min(s, t)$
- The process $\{B_t\}$ is path-continuous

Proof. (\Rightarrow) For all t_1, \dots, t_m , we have

$$(B_{t_m} - B_{t_{m-1}}, \dots, B_{t_2} - B_{t_1}) \equiv N(0, \Sigma)$$

as B_t has stationary, independent increments. Hence, $(B_{t_1}, B_{t_2}, \dots, B_{t_n})$ is normally distributed. Thus B_t is a Gaussian process. We can also show that $\mathbb{E}(B_t) = 0$ and $\mathbb{E}(B_s B_t) = \min(t, s)$.

(\Leftarrow) TO PROVE □

Corollary. *Let B_t be a Brownian motion. The so are the following:*

- $\{B_{t+t_0} - B_{t_0}, t \geq 0\}$
- $\{-B_t, t \geq 0\}$
- $\{cB_{t/c^2}, t \geq 0, c \neq 0\}$
- $\{tB_{1/t}, t \geq 0\}$

Proof. Here, we prove that $\{X_t\} = \{tB_{1/t}\}$ is a Brownian motion. Consider $\sum^n \alpha_i X_{t_i}$. Then by a telescoping argument, we know that the process is Gaussian (can be written as a sum of $X_{t_1}, X_{t_2} - X_{t_1}$, etc). We can also easily show that $\mathbb{E}(X_t) = 0$ and $\mathbb{E}(X_t X_s) = \min(s, t)$ as required.

We now show that $\lim_{t \rightarrow 0} X_t = 0$ a.s. Fix $\epsilon \geq 0$. We must show

$$\mathbb{P} \left(\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{0 < t < \frac{1}{m}} \{|X_t| \leq \frac{1}{n}\} \right) = 1$$

However, as $|X_t|$ has the same distribution as $|B_t|$ (as they are both Gaussian with same mean and covariance), we have that this is equivalent to

$$\mathbb{P} \left(\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{0 < t < \frac{1}{m}} \{|B_t| \leq \frac{1}{n}\} \right)$$

which is clearly one. □

5. LECTURE 5 - TUESDAY 15 MARCH

Theorem 5.1 (Properties of the Brownian motion). *We have*

$$\mathbb{P}(B_t \leq x | B_{t_0} = x_0, \dots, B_{t_n} = x_n) = \mathbb{P}(B_t \leq x | B_s = x_s) = \Phi\left(\frac{x - x_s}{\sqrt{t - s}}\right)$$

Theorem 5.2. *The joint density of $(B_{t_1}, \dots, B_{t_n})$ is given by*

$$g(x_1, \dots, x_n) = \prod_{j=1}^n f(x_{t_j} - x_{t_{j-1}}, t_j - t_1)$$

where $f(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$

Theorem 5.3 (Density and distribution of Brownian bridge). *Let $t_1 < t < t_2$. Then*

$$g(B_t | B_{t_1} = a, B_{t_2} = b) \equiv N \left(a + \frac{(b-a)(t-t_1)}{t_2-t_1}, \frac{(t_2-t)(t-t_1)}{t_2-t_1} \right).$$

The density of $B_t | B_{t_1} = a, B_{t_2} = b$ is given as

$$\frac{g_{t_1, t, t_2}(a, b, t)}{g_{t_1, t_2}(a, b)}$$

Theorem 5.4 (Joint distribution of B_t and B_s). *We have*

$$\begin{aligned} \mathbb{P}(B_s \leq x, B_t \leq y) &= \\ \mathbb{P}(B_s \leq x, B_t - B_s \leq y - B_s) &= \\ &= \int_{-\infty}^x \int_{-\infty}^{y-z} \frac{1}{\sqrt{2\pi s}} e^{-\frac{z_1^2}{2s}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{-z_2^2}{2(t-s)}} dz_1 dz_2 \\ &= \int_{-\infty}^x \int_{-\infty}^y \frac{1}{\sqrt{2\pi s}} e^{-\frac{x_1^2}{2s}} \end{aligned}$$

5.1. Properties of paths of Brownian motion.

Definition 5.5 (Variation). Let g be a real function. Then the variation of g over an interval $[a, b]$ is defined as

$$V([a, b]) = \lim_{\Delta \rightarrow 0} \sum_{j=1}^n |g(t_j) - g(t_{j-1})|$$

where Δ is the size of the partition $a = t_0, \dots, t_n = b$ of $[a, b]$.

Definition 5.6 (Quadratic variation). The quadratic variation of a function g over an interval $[a, b]$ is defined by

$$[g, g] = \lim_{\Delta \rightarrow 0} \sum_{j=1}^n |g(t_j) - g(t_{j-1})|^2$$

Theorem 5.7 (Non-differentiability of Brownian motion). *Paths of Brownian motion are continuous almost everywhere, by definition. Consider now, the differentiability of B_t . We claim that a Brownian motion is non-differentiable almost surely, that is,*

$$\lim_{t \rightarrow s} \frac{|B_t - B_s|}{|t - s|} = \infty \quad a.s.$$

We claim that

- (i) *Brownian motion is not differentiable almost surely for any $t \geq 0$.*
- (ii) *Brownian motion has infinite variation on any interval $[a, b]$, that is,*

$$V_B([a, b]) = \lim_{\Delta \rightarrow 0} \sum_{j=1}^n |B(t_j) - B(t_{j-1})| = \infty \quad a.s.$$

(iii) Brownian motion has quadratic variation t on $[0, t]$, that is

$$[B, B]([0, t]) = \lim_{\Delta \rightarrow 0} \sum_{j=1}^n |B(t_j) - B(t_{j-1})|^2 = t \quad a.s.$$

Proof. We know that $(B_{ct_1}, \dots, B_{ct_n}) \stackrel{d}{=} (c^H B_{t_1}, \dots, c^H B_{t_n})$. This is true as $B_{ct} \equiv N(0, ct) = c^{1/2} N(0, t)$ as required.

Now suppose X_t is H -self-similar with stationary increments for some $0 < H < 1$ with $X_0 = 0$. Then, for any fixed t_0 , we have

$$\lim_{t \rightarrow t_0} \frac{|X_t - X_{t_0}|}{t - t_0} = \infty \quad a.s.$$

Consider

$$\mathbb{P}(\limsup_{t \rightarrow t_0} \frac{|X_t - X_{t_0}|}{t - t_0} \geq M)$$

which by stationary increments, is equal to

$$\begin{aligned} \mathbb{P}(\limsup_{t \rightarrow t_0} \frac{|X_t|}{t} \geq M) &= \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcup_{k \geq n} \frac{|B_{t_n} - B_{t_0}|}{|t_n - t_0|} \geq M\right) \\ &> \lim_{n \rightarrow \infty} \mathbb{P}\left(\frac{|B_{t_n} - B_{t_0}|}{|t_n - t_0|} \geq M\right) \\ &= \lim_{n \rightarrow \infty} \mathbb{P}(|N(0, 1)| \geq M \cdot (t_n - t_0)^{1/2}) \end{aligned}$$

and as the RHS goes to zero, we have

$$\mathbb{P}(|N(0, 1)| \geq M \cdot (t_n - t_0)^{1/2}) \rightarrow 1$$

as required. \square

Now, Assume $V_B([0, t]) < \infty$ almost surely. Consider $Q_n = \sum_{j=1}^n |B_{t_j} - B_{t_{j-1}}|^2$. Then we have

$$Q_n \leq \max_{0 \leq j \leq n} |B_{t_j} - B_{t_{j-1}}| \cdot \sum_{j=1}^n |B_{t_j} - B_{t_{j-1}}|$$

Let $\Delta \rightarrow 0$. Then

$$\begin{aligned} \lim_{\Delta \rightarrow 0} Q_n &\leq \lim_{\Delta \rightarrow 0} \max_{0 \leq j \leq n} |B_{t_j} - B_{t_{j-1}}| \cdot \sum_{j=1}^n |B_{t_j} - B_{t_{j-1}}| \\ &\leq \lim_{\Delta \rightarrow 0} \max_{0 \leq j \leq n} |B_{t_j} - B_{t_{j-1}}| \cdot V_B([0, t]) \\ &\leq 0 \cdot V_B([0, t]) = 0 \end{aligned}$$

because B_s is uniformly continuous on $[0, t]$. This is a contradiction to $V_B([0, t]) < \infty$.

Proof. We now show that $\mathbb{E}((Q_n - t)^2) \rightarrow 0$ as $n \rightarrow \infty$. In fact, we have

$$\begin{aligned}\mathbb{E}(Q_n) &= \sum_{i=1}^n \mathbb{E}(B_{t_j} - B_{t_{j-1}})^2 \\ &= \sum_{j=1}^n (t_j - t_{j-1}) = t\end{aligned}$$

We now show L^2 convergence. We have

$$\begin{aligned}\mathbb{E}(Q_n - t)^2 &= \mathbb{E}((Q_n - \mathbb{E}(Q_n))^2) \\ &= \mathbb{E}\left(\sum_{j=1}^n Y_j\right) \quad \text{where } Y_j = |B_{t_j} - B_{t_{j-1}}|^2 - \mathbb{E}(|B_{t_j} - B_{t_{j-1}}|^2) \\ &= \sum_{j=1}^n \mathbb{E}(Y_j^2) \\ &\leq \sum_{j=1}^n |t_j - t_{j-1}|^2 \cdot \mathbb{E}|N(0, 1)|^4 \\ &\leq C \cdot \Delta \cdot t \rightarrow 0\end{aligned}$$

as $\Delta \rightarrow 0$. Thus we have convergence in L^2 . \square

6. LECTURE 6 - THURSDAY 17 MARCH

Theorem 6.1 (Martingales related to Brownian motion). *Let $B_t, t \geq 0$ be a Brownian motion. Then the following are martingales with respect to $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$.*

- (1) $B_t, t \geq 0$.
- (2) $B_t^2 - t, t \geq 0$.
- (3) For any u , $e^{uB_t - \frac{u^2 t}{2}}$

Proof. (1) is simple.

(2). We know that $\mathbb{E}(|B_t^2|)$ is finite for any t . We can also easily show $\mathbb{E}(B_t^2 - t | \mathcal{F}_s) = B_s^2 - s$ a.s. \square

Theorem 6.2. *Let X_t be a martingale satisfying $X_t^2 - t$ is also a martingale. Then X_t is a Brownian motion.*

Definition 6.3 (Hitting time). Let $T_\alpha = \inf_{\{t \geq 0, B_t = \alpha\}}$

- (1) If $\alpha = 0$, $T_0 = 0$.
- (2) If $\alpha > 0$, then

$$\mathbb{P}(T_\alpha \leq t) = 2\mathbb{P}(B_t \geq \alpha) = \frac{2}{\sqrt{2\pi t}} \int_\alpha^\infty e^{-\frac{x^2}{2t}} dx$$

We clearly have $T_\alpha > t \iff \sup_{0 \leq s \leq t} B_s < \alpha$

7. LECTURE 7 - TUESDAY 22 MARCH

Theorem 7.1 (Arcsine law). *Let B_t be a Brownian motion. Then*

$$\mathbb{P}(B_t = 0, \text{ for at least once, } t \in [a, b]) = \frac{2}{\pi} \arccos \sqrt{\frac{a}{b}}$$

Example 7.2. Processes derived from Brownian motion

(1) Brownian bridge

$$X_t = B_t - tB_1 \quad t \in [0, 1]$$

Consider $X \equiv F(x)$, with X_1, \dots, X_n data. Our empirical distribution

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{X_j \leq x\}}$$

We can then prove that

$$n \sqrt{F_n(x) - F(x)} \rightarrow X_t \quad t \in (0, 1)$$

(2) Diffusion process

$$X_t = \mu t + \sigma B_t$$

This is a Gaussian process with $\mathbb{E}(X_t) = \mu t$, $\text{Cov}(X_t, X_s) = \sigma^2 \min(s, t)$.

(3) Geometric Brownian motion

$$X_t = X_0 e^{\mu t + \sigma B_t}$$

This is not a Gaussian process.

(4) Higher dimensional Brownian motion

$$B_t = (B_t^1, \dots, B_t^n)$$

where the B^i are independent Brownian motions, then

7.1. Construction of Brownian motion. Define a stochastic process

$$\hat{B}_t^n = \frac{S_{[nt]} - \mathbb{E}(S_{[nt]})}{\sqrt{n}}$$

With

$$\tilde{B}_t^n = \begin{cases} \hat{B}_t^n & \text{if } t = \frac{i}{n} \\ 0 & \text{otherwise} \end{cases}$$

Then we can prove that

$$\tilde{B}_t^n \Rightarrow B_t \quad \text{on } [0, 1]$$

Definition 7.3 (Stochastic integral). We now turn to defining expressions of the form

$$\int_0^A X_t dY_t$$

with X_t, Y_t stochastic processes.

Definition 7.4 ($\int_0^t f(B_s) ds$). We have

$$\int_0^t f(t) dt$$

exists if f is bounded and continuous, except on a set of Lebesgue measure zero. Thus, we can set

$$\int_0^t f(t) dt = \lim_{\Delta \rightarrow 0} \sum_{j=1}^n f(B_{y_j})(t_j - t_{j-1})$$

if $f(x)$ is bounded.

We now seek to find $\int_0^1 B_s ds$. Consider $Q_n = \sum_{j=1}^n B_{y_j}(t_j - t_{j-1})$. As the sum of normal variables, we know that $Q_n \equiv N(\mu_n, \sigma_n^2)$.

Since $\int_0^1 B_s ds$ is normally distributed with mean 0, we have

$$\begin{aligned} \mathbb{E}(X^2) &= \mathbb{E} \int_0^1 B_s dx \int_0^1 B_t dt \\ &= \int_0^1 \mathbb{E}(B_s B_t) ds dt \\ &= \int_0^1 \min(s, t) ds dt \\ &= 1/3 \end{aligned}$$

8. LECTURE 8 - THURSDAY 24 MARCH

$$\int_0^1 f(B_s) ds = \lim_{\Delta \rightarrow 0} \sum_{j=1}^n f(B_{y_j})(t_{j+1} - t_j)$$

Recall

$$\int_0^1 B_s ds \sim N(0, \frac{1}{3}) = \lim_{\Delta \rightarrow 0} \sum_{i=1}^n B_{y_j}(t_{j+1} - t_j)$$

Consider $I = \int X_s dY_s$. We have the following:

Theorem 8.1. *I exists if*

- (1) *The functions f, g are not discontinuous at the same point x .*
- (2) *f is continuous and g has bounded variation or,*
- (2)' *f has finite p -variation and g has finite q -variation, where*

$$1/p + 1/q = 1$$

For any p , we define p -variation by

$$\lim_{\Delta \rightarrow 0} \sum_{j=1}^n |f(t_j) - f(t_{j-1})|^p$$

Theorem 8.2. If $J = \int_0^1 f(t) dg(t)$ exists for any continuous f , then g must have finite variation.

Theorem 8.3. B_s has bounded p variation for any $p \geq 2$ and unbounded q -variation for any $q < 2$.

Proof. We can write (for $p \geq 2$), $p = 2 + (p - 2)$. Then we have

$$\Delta_t = \lim_{\Delta \rightarrow 0} \max |B_{t_j} - B_{t_{j-1}}|^{p-2} \sum_{j=1}^n |B_{t_j} - B_{t_{j-1}}|^2$$

and hence Δ_t exists. □

Corollary. Thus $\int_0^1 dB_t$ is well defined if f has finite variation (as setting $q = 1$, $p \geq 2$ gives $1/p + 1/q > 1$).

Consider $\int_0^1 B_s dB_s$ - is this an R-S integral? Consider

$$\Delta_{1n} = \sum_{j=1}^{\infty} B_{t_j} (B_{t_{j+1}} - B_{t_j}), \Delta_{2n} = \sum_{j=1}^{\infty} B_{t_{j+1}} (B_{t_{j+1}} - B_{t_j})$$

We have that

$$\Delta_{2n} - \Delta_{1n} = \sum_{j=1}^n (B_{t_{j+1}} - B_{t_j})^2 \rightarrow 1$$

and

$$\Delta_{1n} + \Delta_{2n} = \sum_{j=1}^n (B_{t_{j+1}}^2 - B_{t_j}^2) = B_1^2.$$

Thus we know

$$\begin{aligned} \Delta_{2n} &\rightarrow \frac{1}{2}(B_1^2 + 1) \\ \Delta_{1n} &\rightarrow \frac{1}{2}(B_1^2 - 1) \end{aligned}$$

Definition 8.4 (Itô integral). The Itô integral is defined by evaluating $f(B_{t_j})$, the left-hand endpoint at each partition interval $[t_j, t_{j+1})$

9. LECTURE 9 - TUESDAY 29 MARCH

Definition 9.1 (Itô integral). Consider $\int_0^1 f(s) dB_s$. Where $f(s)$ is a real function, B_s a Brownian motion. We define the integral in two steps.

(1) If $f(s)$ is a step function, define

$$\begin{aligned} I(f) &= \int_0^1 f(s) dB_s = \sum_{j=1}^m \int_{t_j}^{t_{j+1}} f(s) dB_s \\ &= \sum_{j=1}^m c_j (B_{t_{j+1}} - B_{t_j}) \end{aligned}$$

(2) If $f(s) \in L^2([0, 1])$, then let f_n be a sequence in $L^2([0, 1])$ such that $f_n \rightarrow f$ in $L^2([0, 1])$.

Then define $I(f)$ to be the limitation in such situations such that

$$\mathbb{E}(I(f_n) - I(f))^2 \rightarrow 0$$

in $L^2([0, 1])$ or Let $I(f) = \lim_{n \rightarrow \infty} I(f_n)$ in probability.

Remark. If $f(x), g(x)$ are given step functions then $\alpha I(f) + \beta I(g) = I(\alpha f + \beta g)$.

Remark. If $f(x)$ is a step function, then $I(f) \sim N(0, \int_0^1 f^2(s) ds)$

Proof.

$$I(f) = \sum_{j=1}^m c_j (B_{t_{j+1}} - B_{t_j}) \sim N(0, \sigma^2)$$

where $\sigma^2 = \sum_{j=1}^m c_j^2 (t_{j+1} - t_j)$. □

Theorem 9.2. $I(f)$ is well defined (independent of the choice of f_n .)

Proof. Let $f_n, g_n \rightarrow f$ in $L^2([0, 1])$. We then need to only compute

$$\Delta_{n,m} = \mathbb{E}(I(f_n) - I(g_m))^2 \rightarrow 0$$

as $m, n \rightarrow \infty$.

In fact,

$$\begin{aligned} \Delta_{n,m} &= \mathbb{E}(I(f_n - g_m))^2 = \int_0^1 (f_n - g_m)^2 dx \\ &\leq 2 \int_0^1 (f_n - f)^2 dx + 2 \int_0^1 (g_m - f)^2 dx \\ &\rightarrow 0 \end{aligned}$$

as $n, m \rightarrow \infty$. □

Remark. If f is continuous bounded variation, then

$$\begin{aligned} \int_0^1 f(s) dB_s &= (R.S.) \int_0^1 f(s) dB_s \\ &= \lim_{\delta \rightarrow 0} \sum_{j=1}^n f(t_j) (B_{t_{j+1}} - B_{t_j}) \end{aligned}$$

Proof. Case 1).

$$\alpha I(f) + \beta I(g) = \lim_{n \rightarrow \infty} [\alpha I(f_n) + \beta I(g_n)] = \lim_{n \rightarrow \infty} [I(\alpha f_n + \beta g_n)] = I(\alpha f + \beta g)$$

and thus $\alpha f_n + \beta g_n \rightarrow \alpha f + \beta g \in L^2([0, 1])$. \square

Case 2). If $I(f) = \lim_{\delta \rightarrow 0} I(f_n)$ in probability. Then

$$\begin{aligned} I(f_n) &\sim N(0, \sigma_n^2) \\ &\sim N(0, \int_0^1 f_n^2(s) ds) \\ &\rightarrow N(0, \int_0^1 f^2(s) ds) \end{aligned}$$

if $\sigma_n^2 \rightarrow \int_0^1 f^2(x) dx$. In fact, as

$$\begin{aligned} \int_0^1 f_n^2 dx &= \int_0^1 (f_n - f + f)^2 dx \\ &= \int_0^1 f^2 dx + \int_0^1 (f_n - f)^2 dx + \int_0^1 (f_n - f)f dx \\ &\rightarrow \int_0^1 f^2 dx \end{aligned}$$

as other terms tend to zero by L^2 convergence and Hölder's inequality.

Remark. (R.S.) $\int_0^1 f(s) dB_s$ exists if f is of bounded variation.

Remark. If f is continuous then $\int_0^1 f^2 dx < \infty$ and

$$f_n(t) = \sum_{j=1}^n f(t_j) I_{[t_j, t_{j+1})} \rightarrow f(t) \text{ in } L^2([0, 1])$$

Thus,

$$\begin{aligned} I(f) &= \lim_{\delta \rightarrow 0} I(f_n) \\ &= \lim_{\delta \rightarrow 0} \sum_{j=1}^m f(t_j) (B_{t_{j+1}} - B_{t_j}) \end{aligned}$$

We have to prove

$$\int_0^1 (f_n - f)^2 dx \rightarrow 0$$

if f is continuous.

10. LECTURE 10 - THURSDAY 31 MARCH

We have

$$(\text{It}\hat{o}) \int_0^1 f(s) dB_s \sim N(0, \int_0^1 f(t)^2 dt)$$

if f is continuous, and of finite variation. In this case, we can write

$$(\text{R.S.}) \int_0^1 f(s) dB_s = (\text{It}\hat{o}) \int_0^1 f(s) dB_s = \lim_{\Delta \rightarrow 0} \sum_{j=1}^n f(t_j)(B_{t_{j+1}} - B_{t_j})$$

Example 10.1. We have

$$\int_0^1 (1-t) dB_t = (\text{It}\hat{o}) \int_0^1 (1-t) dB_t \sim N(0, \int_0^1 (1-t)^2 dt) = N(0, \frac{1}{3})$$

and by integrating by parts, we have

$$\int_0^1 (1-t) dB_t = (1-t)B_t|_0^1 - \int_0^1 B_t d(1-t) = 0 + \int_0^1 B_t dt \sim N(0, \frac{1}{3})$$

Now consider

$$(\text{It}\hat{o}) \int_0^1 X_s dB_s$$

where we now allow X_s to be a stochastic process.

Write $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$. Let π be the collection of X_s such that

- (1) X_s is adapted to \mathcal{F}_s , that is, for any s , X_s is \mathcal{F}_s measurable.
- (2) $\int_0^1 X_s^2 ds < \infty$ almost surely (R.S)

Let $\pi' = \{X_s \mid X_s \in \pi, \int_0^1 \mathbb{E}(X_s^2) < \infty\}$. Then $\pi' \subset \pi$. Let $X_s = e^{B_s^2}$. Then

$$\mathbb{E}(X_s^2) = \mathbb{E}(e^{B_s^2}) = \begin{cases} \frac{1}{\sqrt{1-4s}} & 0 \leq s < \frac{1}{4} \\ \infty & s \geq \frac{1}{4} \end{cases}$$

Definition 10.2 (Itô integral for stochastic integrands). We proceed in two steps.

- (1) Let $X_s = \sum_{j=1}^n \zeta_j \mathbf{1}_{[t_j, t_{j+1})}$ where ζ_j is \mathcal{F}_{t_j} measurable. Then

$$I(X) = \sum_{j=1}^n \zeta_j (B_{t_{j+1}} - B_{t_j}).$$

- (2) If $X \in \pi$, there exists a sequence $X^n \in \pi'$ such that X^n are step process with

$$\int_0^1 |X_s^n - X_s|^2 ds \rightarrow 0$$

as $n \rightarrow \infty$ in probability or in $L^2([0, 1])$ if $X_s \in \pi'$.

Proof. We show only for X_s continuous, the general case can be found in Hu-Hsing Kuo (p.65). As X_s is continuous, then it is in π . Then choose

$$X_s^n = X_0 + \sum_{j=1}^n j = 1^n X_{t_j} \mathbf{1}_{[t_j, t_{j+1})}$$

Then $X_s \in \pi'$ and

$$\mathbb{E}(|X_s^n - X_s|^2) \rightarrow 0$$

for any $s \in (0, 1)$.

We can also show that

$$\mathbb{E}(|X_s^n - X_s|^2) < \infty$$

and so by the dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} \int_0^1 \mathbb{E}(|X_s^n - X_s|^2) ds = 0.$$

□

Finally if $X_s \in \pi$, the Itô integral $\int_0^1 X_s dB_s$ is defined as

$$I(X) = \lim_{\delta \rightarrow 0} I(X^n)$$

in probability or in L^2 if $X \in \pi'$.

11. LECTURE 11 - TUESDAY 5 MARCH

Let $I(X) = \int_0^1 X_s dB_s$ in the Itô sense. We then require

- (1) X_s is $\mathcal{F}_s = \sigma\{B_t, 0 \leq t \leq s\}$ -measurable
- (2) $\int_0^1 X_s^2 ds < \infty$ almost surely.

Then $I(X) = \lim_{n \rightarrow \infty} I(X_n)$ where X_n is a sequence of step functions converging to X in L^2 , that is,

$$\int_0^1 (X_s^n - X_s)^2 ds \rightarrow 0$$

We then show that this definition is independent of the sequence of step functions. For any Y_s step process, we have

$$\int_0^1 (Y_s^n - Y_s^m)^2 ds \leq 2 \int_0^1 (Y_s^m - X_s)^2 ds + 2 \int_0^1 (X_s^n - X_s)^2 ds \rightarrow 0$$

Theorem 11.1 (Properties of the Itô integral).

- (1) For any $\alpha, \beta \in \mathbb{R}$, $X, Y \in \pi$,

$$I(\alpha X + \beta Y) = \alpha I(X) + \beta I(Y)$$

(2) For any $X_s \in \pi'$, we have

$$\mathbb{E}(I(X)) = 0, \quad \mathbb{E}(I^2(X)) = \int_0^1 \mathbb{E}(X_s^2) ds$$

If $X' \in \pi', Y_s \in \pi'$, then

$$\mathbb{E}(I(X)I(Y)) = \int_0^1 \mathbb{E}(X_s Y_s) ds$$

(3) If X_s continuous then

$$I(X) = \lim_{\delta \rightarrow 0} \sum_{j=1}^n X_{t_j} (B_{t_{j+1}} - B_{t_j})$$

in probability.

(4) If X is continuous and of finite variation and $\int_0^1 B_s dX_s < \infty$ then

$$(R.S.) \int_0^1 = (It\hat{o}) \int_0^1 X_s dB_s = X_1 B_1 - X_0 B_0 - \int_0^1 B_s dX_s$$

Proposition 11.2. We now show why we require

- (1) X_s is $\mathcal{F}_s = \sigma\{B_t, 0 \leq t \leq s\}$ -measurable
- (2) $\int_0^1 X_s^2 ds < \infty$ almost surely.

Proof. Motivation for (1)

$$\begin{aligned} X_s &= \sum_{j=1}^n \zeta_j \mathbf{1}_{(t_j, t_{j+1})} \\ I(X) &= \sum_{j=1}^n \zeta_j (B_{t_{j+1}} - B_{t_j}) \\ \mathbb{E}(I(X)) &= \sum_{j=1}^n \mathbb{E}(\zeta_j (B_{t_{j+1}} - B_{t_j})) \\ &= \sum_{j=1}^n \mathbb{E}(\mathbb{E}(\zeta_j (B_{t_{j+1}} - B_{t_j}) | \mathcal{F}_{t_j})) \\ &= \sum_{j=1}^n \mathbb{E}(\zeta_j \mathbb{E}(B_{t_{j+1}} - B_{t_j} | \mathcal{F}_{t_j})) \end{aligned}$$

Motivation for (2)

$$\begin{aligned}
\mathbb{E}(I^2(X)) &= \mathbb{E}\left(\sum_{j=1}^n \zeta_j(B_{t_{j+1}} - B_{t_j})\right)^2 \\
&= 2 \sum_{i < j} \mathbb{E}(\mathbb{E}(\zeta_i \zeta_j (B_{t_{j+1}} - B_{t_j})(B_{t_{i+1}} - B_{t_i}) | \mathcal{F}_{t_i}) + \sum_{j=1}^n \mathbb{E}(\zeta_j^2 (B_{t_{j+1}} - B_{t_j})^2) \\
&= \sum_{j=1}^n \mathbb{E}(\zeta_j^2)(t_{j+1} - t_j) \\
&= \int_0^1 \mathbb{E}(X_t^2) dt
\end{aligned}$$

We now show that there $X_s^n \in \pi'$. such that $\int_0^1 \mathbb{E}(X_s^n - X_s)^2 ds \rightarrow 0$. We have

$$\begin{aligned}
\mathbb{E}(I^2(X)) &= \mathbb{E}(I(X)) - I(X^n) + \mathbb{E}(I(X_s^n))^2 \\
&= \mathbb{E}(I^2(X^n)) + 2\mathbb{E}(I(X^n)(I(X) - I(X^n))) + \mathbb{E}(I(X) - I(X^n))^2
\end{aligned}$$

By Cauchy-Swartz, the middle term tends to zero, and by definition, the third term tends to zero. Thus we have

$$\lim_{n \rightarrow \infty} \mathbb{E}(I^2(X_n)) = \mathbb{E}(I^2(X))$$

Let X_s^n be step processes. We now show

$$\mathbb{E}(X_s^n - X_s)^2 \rightarrow 0 \Rightarrow \int_0^1 \mathbb{E}(X_s^n - X_s)^2 ds \rightarrow 0$$

By definition, we have

$$I(X) = \lim_{n \rightarrow \infty} I(X^n) = \lim_{\delta \rightarrow 0} \sum_{j=1}^n X_{t_j} (B_{t_{j+1}} - B_{t_j}).$$

and we proved the required result in lectures.

We now show that the (R.S.) integral exists if X is continuous and of finite variation, and $\int_0^1 B_s dX_s < \infty$, the our integration by parts formula holds.

We have

$$\begin{aligned}
(R.S.) \int_0^1 X_s dB_s &= \lim_{\delta \rightarrow 0} \sum_{j=1}^n X_{y_j} (B_{t_{j+1}} - B_{t_j}) \\
&= \lim_{\delta \rightarrow 0} \sum_{j=1}^n X_{t_j} (B_{t_{j+1}} - B_{t_j})
\end{aligned}$$

which is our Itô integral by definition. □

Definition 11.3 (Itô process). Suppose $Y_t = \int_0^t X_s dB_s$, $t \geq 0$ is well defined, for $X_s \in \pi$. Then Y_t is an Itô processes. To show a process Y_t is an Itô process, we need to show that $\int_0^t |X_s| ds) M \infty$ a.s. and $\int_0^t |X_s|^2 ds < \infty$ a.s.

Theorem 11.4. We have that Y_t is continuous (except on a null set), is of infinite variation, as

$$\begin{aligned} \sum_{j=1}^n |Y_{t_{j+1}} - Y_{t_j}| &= \sum_{j=1}^n \left| \int_{t_j}^{t_{j+1}} X_s dB_s \right| \\ &\geq \sum_{j=1}^n \min_s |X_s| |B_{t_{j+1}} - B_{t_j}| \\ &\geq C \sum_{j=1}^n |B_{t_{j+1}} - B_{t_j}| \end{aligned}$$

12. LECTURE 12 - THURSDAY 7 MARCH

From before, consider the Itô process $Y_t = \int_0^t X_s dB_s$.

Lemma 12.1. $\mathbb{E}(\int_s^t X_u dB_u | \mathcal{F}_s) = 0$.

Proof. Let $X_u = \sum_{j=1}^n \zeta_j \mathbf{1}_{[t_j, t_{j+1})}$. Then

$$\begin{aligned} \mathbb{E}\left(\int_s^t X_u dB_u \mid \mathcal{F}_s\right) &= \sum_{j=1}^n \mathbb{E}(\zeta_j (B_{t_{j+1}} - B_{t_j}) \mid \mathcal{F}_s) \\ &= \sum_{j=1}^n \mathbb{E}(\mathbb{E}(\zeta_j (B_{t_{j+1}} - B_{t_j}) \mid \mathcal{F}_s) \mid \mathcal{F}_{t_j}) \\ &= \sum_{j=1}^n \mathbb{E}(\mathbb{E}(\zeta_j (B_{t_{j+1}} - B_{t_j}) \mid \mathcal{F}_{t_j}) \mid \mathcal{F}_s) \\ &= 0 \end{aligned}$$

where the third equality follows from the fact that (\mathcal{F}_t) is an increasing sequence of σ -fields and the final follows from the fact that $B_{t_{j+1}} - B_{t_j}$ is independent of \mathcal{F}_{t_j} . \square

Definition 12.2 (Local martingale). A process Y_t is a local martingale if there exists a sequence of stopping times $\tau_n, n \geq 1$ such that $Y_{\min(t, \tau_n)}$ is a martingale.

Proposition 12.3. We have the following.

- (1) If $X_s \in \pi'$, that is, $\int_0^t \mathbb{E}(X_s^2) ds < \infty$, then (Y_t, \mathcal{F}_t) is a martingale.
- (2) If $X_s \in \pi$, that is $\int_0^t X_s^2 ds < \infty$ a.s., then (Y_t, \mathcal{F}_t) is a local martingale.
- (3) For $f(x)$ satisfying $\int_0^t f^2(z) dz < \infty$, we have

$$Y_t = \int_0^t f(s) dB_s$$

is a Gaussian process.

Proof. We have Y_t is trivially \mathcal{F}_t -measurable.

We have $\mathbb{E}(|Y_t|) < \infty$ a.s. as $\mathbb{E}(Y_t^2) = \int_0^t \mathbb{E}(X_s^2) ds < \infty$ by assumption.

We have

$$\begin{aligned}\mathbb{E}(Y_t | \mathcal{F}_s) &= \mathbb{E}\left(\int_0^s X_u dB_u + \int_s^t X_u dB_u \mid \mathcal{F}_s\right) \\ &= \mathbb{E}(Y_s | \mathcal{F}_s) + \mathbb{E}\left(\int_s^t X_u dB_u \mid \mathcal{F}_s\right) \\ &= Y_s\end{aligned}$$

from the previous lemma.

Now assuming $X_s \in \pi'$, there exists an X_s^n a step process such that

$$\int_0^t \mathbb{E}(X_u^n - X_u)^2 du \rightarrow 0$$

as $n \rightarrow \infty$.

Set $Y_t^n = \int_0^t X_u^n dB_u$. Then

$$\begin{aligned}\int_s^t X_u dB_u &= Y_t - Y_s \\ &= Y_t - Y_t^n + Y_t^n - Y_s^n + Y_s^n - Y_s\end{aligned}$$

Then for each n , we have

$$Z = \mathbb{E}\left(\int_s^t X_u dB_u\right) = \mathbb{E}(Y_t - Y_t^n | \mathcal{F}_s) + \mathbb{E}(Y_s^n - Y_s | \mathcal{F}_s)$$

We only need to prove $\mathbb{E}(|Z|^2) \rightarrow 0$ which implies $\mathbb{E}(Z^2) = 0$ and thus $Z = 0$ almost surely. We have

$$\mathbb{E}(Z^2) \leq 2\mathbb{E}(Y_t - Y_t^n)^2 + 2\mathbb{E}(Y_s^n - Y_s)^2 \rightarrow 0$$

by definition of Y_t^n .

□

13. LECTURE 13 - TUESDAY 12 MARCH

Theorem 13.1. *Let f be continuous. Then*

$$\lim_{\delta \rightarrow 0} \sum_{j=1}^n f(\theta_j)(B_{t_{j+1}} - B_{t_j})^2 = \int_0^t f(B_s) ds$$

Proof. Let

$$Q_n = \sum_{j=1}^n |f(\theta_j) - f(B_{t_j})| |B_{t_{j+1}} - B_{t_j}|^2$$

Note that $\sum_{j=1}^n f(B_{t_j})(B_{t_{j+1}} - B_{t_j})^2 \rightarrow \int_0^t f(B_s) ds$ in probability.

It only needs to show that $Q_n \rightarrow 0$ in probability. We have

$$Q_n \leq \max_{1 \leq j \leq n} |f(\theta_j) - f(B_{t_j})| \sum_{j=1}^n |B_{t_{j+1}} - B_{t_j}|^2 \rightarrow 0 \cdot t = 0$$

in probability from the quadratic variance of the brownian motion □

Theorem 13.2. *Let f be bounded on $[0, 1]$. Then*

$$\lim_{\delta \rightarrow 0} \sum_{j=1}^n f(X_{t_j})(B_{t_{j+1}} - B_{t_j})^2 = \int_0^t f(X_s) ds$$

Proof. We only prove for cases where $\int_0^t \mathbb{E}(X_s^2) ds$ is finite. In this case, there exists X_s^n step process such that

$$\int_0^t \mathbb{E}(X_s^n - X_s)^2 ds \rightarrow 0$$

as $n \rightarrow \infty$.

In fact, we can let $X_s^n = \sum_{j=1}^n X_{t_j} \mathbf{1}_{[t_j, t_{j+1}]}$. Let

$$Y_t^n = \int_0^t X_s^n dB_s$$

Then $Y_{t_{j+1}}^n - Y_{t_j}^n = \int_{t_j}^{t_{j+1}} X_s^n dB_s$.

Therefore,

$$\sum_{j=1}^n (Y_{t_{j+1}} - Y_{t_j})^2 = \sum_{j=1}^n (Y_{t_{j+1}} - Y_{t_j}^n + Z_{1j} + Z_{2j})^2$$

where $Z_{1j} = Y_{t_{j+1}} - Y_{t_{j+1}}^n$ and $Z_{2j} = Y_{t_j} - Y_{t_j}^n$. Continuing, we have

$$\begin{aligned} &= \sum_{j=1}^n (Y_{t_{j+1}} - Y_{t_j}^n)^2 + \text{error} \\ &= \sum_{j=1}^n X_{t_j}^2 (B_{t_{j+1}} - B_{t_j})^2 + \text{error} \\ &\rightarrow \int_0^t X_s^2 ds \end{aligned}$$

if the error term goes to zero. We have

$$\begin{aligned} R_n &\leq 2 \sum Z_{1j}^2 + 2 \sum Z_{2j}^2 + 2 \sum |Z_{1j}| \cdot |Y_{t_{j+1}}^n - Y_{t_j}^n| + 2 \sum |Z_{2j}| \cdot |Y_{t_{j+1}}^n - Y_{t_j}^n| + 2 \sum |Z_{1j}| \cdot |Z_{2j}| \\ &\leq \dots + 2(\sum Z_{1j}^2)^{1/2} A_n^{1/2} + 2(\sum Z_{2j}^2)^{1/2} A_n^{1/2} + 2(\sum Z_{1j}^2)^{1/2} \cdot (\sum Z_{2j}^2)^{1/2} \end{aligned}$$

and as $\sum Z_{ij}^2 \rightarrow 0$ in probability, we have our result. \square

Theorem 13.3 (Itô's first formula). *If $f(x)$ is a twice-differentiable function then for any t ,*

$$f(B_t) - f(B_s) = \int_s^t f'(B_u) dB_u + \frac{1}{2} \int_s^t f''(B_u) du$$

Proof. Let $s = t_1, \dots, t_n = t$. We have

$$f(B_t) - f(B_s) = \sum_{j=1}^n [f(B_{t_{j+1}}) - f(B_{t_j})]$$

Applying Taylor's expansion to $f(B_{t_{j+1}}) - f(B_{t_j})$ we get

$$f(B_{t_{j+1}}) - f(B_{t_j}) = f'(B_{t_j})(B_{t_{j+1}} - B_{t_j}) + \frac{1}{2} f''(\theta_j)(B_{t_{j+1}} - B_{t_j})^2$$

and so

$$\begin{aligned} f(B_t) - f(B_s) &= \lim_{\delta \rightarrow 0} \sum_{j=1}^n f'(B_{t_j})(B_{t_{j+1}} - B_{t_j}) + \lim_{\delta \rightarrow 0} \sum_{j=1}^n \frac{1}{2} f''(\theta_j)(B_{t_{j+1}} - B_{t_j})^2 \\ &= \int_s^t f'(B_u) dB_u + \frac{1}{2} \int_s^t f''(B_u) du \end{aligned}$$

\square

Example 13.4.

14. LECTURE 14 - THURSDAY 14 MARCH

Definition 14.1 (Covariation). The covariation of two stochastic processes X_t, Y_t is defined as

$$[X, Y](t) = \frac{1}{4}([X + Y, X + Y](t) - [X - Y, X - Y](t))$$

where $[\cdot, \cdot]$ is the quadratic variation previously defined.

Definition 14.2 (Stochastic differential equation). Let X_t be an Itô process. Then

$$X_t = X_a + \int_a^t \mu(s) ds + \int_a^t \sigma(s) dB_s$$

We write

$$dX_t = \mu(t) dt + \sigma(t) dB_t$$

By convention, we write $dX_t \cdot dY_t = d[X, Y](t)$. In particular, $(dY_t)^2 = d[Y, Y](t)$.

Theorem 14.3. *Let Y_t be path continuous, and let X_t have finite variation. Then*

$$[X, Y](t) = 0.$$

Proof. We have

$$\begin{aligned}
[X, Y](t) &= \frac{1}{4}([X + Y, X + Y] - [X - Y, X - Y]) \\
&= \lim_{\delta \rightarrow 0} \sum_{j=1}^n (X_{t_{j+1}} - X_{t_j})(Y_{t_{j+1}} - Y_{t_j}) \\
&\leq \lim_{\delta \rightarrow 0} \max |Y_{t_{j+1}} - Y_{t_j}| \sum_{j=1}^n |X_{t_{j+1}} - X_{t_j}| \\
&\rightarrow 0
\end{aligned}$$

as Y_t is path continuous and X_t has finite variation. □

Corollary. *From this theorem, we then have*

$$dB_t \cdot dt = 0, \quad (dt)^2 = 0, \quad (dB_t)^2 = dt$$

Corollary. *For an Itô process X_t given above, we then have*

$$d[X, X](t) = dX_t \cdot dX_t = (\mu(t)dt + \sigma(t) dB_t)^2 = \sigma^2(t) dt$$

Corollary. *If $f(x)$ has a twice continuous derivative, then*

$$[f(B_t), B_t](t) = \int_0^t f'(B_s) ds$$

Proof. From Itô's formula, we have

$$df(B_t) = f'(B_t) dB_t + \frac{1}{2} f''(B_t) dt$$

Then from above, we have

$$d[f(B_t), B_t](t) = df(B_t) \cdot dB_t = f'(B_t) dt$$

□

Theorem 14.4 (Itô's lemma). *By Taylor's theorem, we have*

$$\begin{aligned}
df(t, X_t) &= \frac{\partial f(t, X_t)}{\partial t} dt + \frac{\partial f(t, x)}{\partial x} \Big|_{x=X_t} \cdot dX_t + \frac{1}{2} \frac{\partial^2 f(t, x)}{\partial x^2} \Big|_{x=X_t} \cdot (dX_t)^2 \\
&\quad + \frac{1}{2} \frac{\partial^2 f(t, x)}{\partial x \partial t} \Big|_{x=X_t} \cdot dX_t \cdot dt \\
&= \frac{\partial f(t, X_t)}{\partial t} dt + \frac{\partial f(t, x)}{\partial x} \Big|_{x=X_t} \cdot (\mu(t) dt + \sigma(t) dB_t) + \frac{1}{2} \frac{\partial^2 f(t, x)}{\partial x^2} \Big|_{x=X_t} \cdot \sigma^2 dt \\
&= \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \mu(t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \sigma^2(t) \right) dt + \frac{\partial f}{\partial x} \sigma(t) dB_t
\end{aligned}$$

Example 14.5. Let $f(B_t) = e^{B_t}$. Then

$$df(B_t) = e^{B_t} dB_t + \frac{1}{2} e^{B_t} dt$$

Theorem 14.6. Assume $\int_0^T f^2(t) dt < \infty$. Let $X_t = \int_0^t f(s) dB_s - \frac{1}{2} \int_0^t f^2(s) ds$. Then

$$Y_t = e^{X_t}$$

is a martingale with respect to $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$.

Proof. We have

$$\begin{aligned} dY_t &= e^{X_t} dX_t + \frac{1}{2} e^{X_t} (dX_t)^2 \\ &= e^{X_t} [f(t) dB_t - \frac{1}{2} f^2(t) dt] + \frac{1}{2} e^{X_t} f^2(t) dt \\ &= e^{X_t} f(t) dB_t \end{aligned}$$

and thus $Y_t = \int_0^t e^{X_s} f(s) dB_s$ which is a martingale from previous work. \square

15. LECTURE 15 - TUESDAY 19 MARCH

Theorem 15.1 (Multivariate Itô's formula). Let $B_j(t)$ be a sequence of independent Brownian motions. Consider the Itô processes X_t^i , with

$$dX_t^i = b_i(t) dt + \sum_{k=1}^m \sigma_{ki}(t) dB_k(t)$$

Suppose $f(t, x_1, \dots, x_n)$ is a continuous function of its components and has continuous partial derivatives $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial^2 x_i \partial x_j}$. Then

$$df(t, X_t^1, \dots, X_t^m) = \frac{\partial f}{\partial t} dt + \sum_{i=1}^m \frac{\partial f}{\partial x_i} dX_t^i + \frac{1}{2} \sum_{i,j=1}^m \frac{\partial^2 f}{\partial x_i \partial x_j} dX_t^i dX_t^j$$

We have

$$\begin{aligned} dX_t^i dX_t^j &= \sum_{k,s=1}^m \sigma_{ki} \sigma_{sj} dB_k(t) dB_s(t) \\ &= \sum_{k=1}^n \sigma_{ki} \sigma_{kj} dt + \end{aligned}$$

as $d[B_i, B_j](t) = 0$ when B_i, B_j are independent

Example 15.2. Let

$$\begin{aligned} dX_t &= \mu_1(t)dt + \sigma_1(t) dB_1(t) \\ dY_t &= \mu_2(t)dt + \sigma_2(t) dB_2(t) \end{aligned}$$

with B_1, B_2 independent. Then

$$d(X_t \cdot Y_t) = Y_t dX_t + X_t dY_t + d[X, Y](t)$$

and by independence, $d[X, Y](t) = dX_t \cdot dY_t = 0$.

Example 15.3. Let

$$dX_t = \mu_1(t)dt + \sigma_1(t) dB_1(t)$$

$$dY_t = \mu_2(t)dt + \sigma_2(t) dB_1(t).$$

Then

$$d(X_t \cdot Y_t) = Y_t dX_t + X_t dY_t + \sigma_1(t)\sigma_2(t) dt.$$

In particular, if $\mu_1 = \mu_2 = 0$, then

$$X_t Y_t = \int_0^t [\sigma_1(s)Y_s + \sigma_2(s)X_s] dB_s + \int_0^t \sigma_1(s)\sigma_2(s) dt$$

Thus, $Z_t = X_t Y_t - \int_0^t \sigma_1(s)\sigma_2(s) dt$ is a martingale.

Theorem 15.4 (Tanaka's formula). *We have*

$$|B_t - a| = |a| + \int_0^t \text{sgn}(B_s - a) dB_s + \mathcal{L}(t, a)$$

where

$$\mathcal{L}(t, a) = \lim_{\epsilon > 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{|B_s - a| \leq \epsilon} ds$$

Proof. Let

$$f_\epsilon(x) = \begin{cases} |x - a| - \frac{\epsilon}{2} & |x - a| > \epsilon \\ \frac{1}{2\epsilon}(x - a)^2 & |x - a| \leq \epsilon \end{cases}$$

Then we have

$$f'_\epsilon(x) = \begin{cases} 1 & x > a + \epsilon \\ \frac{1}{\epsilon}(x - a) & |x - a| \leq \epsilon \\ -1 & x < a - \epsilon \end{cases}$$

and

$$f''_\epsilon(x) = \begin{cases} 0 & |x - a| > \epsilon \\ \frac{1}{\epsilon} & |x - a| \leq \epsilon \end{cases}$$

Then by Itô's formula, we have

$$f_\epsilon(B_t) = f_\epsilon(0) + \int_0^t f'_\epsilon(B - s) dB_s + \frac{1}{2} \int_0^t f''_\epsilon(B_s) ds$$

Obviously

$$\lim_{\epsilon \rightarrow 0} f_\epsilon(0) = |a|$$

and

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2} \int_0^t f_\epsilon''(B_s) ds = \frac{1}{\epsilon} \int_0^t \mathbf{1}_{|B_s - a| \leq \epsilon} ds = \mathcal{L}(t, a)$$

Note that

$$\int_0^t |f_\epsilon'(B_s) - \text{sgn}(B_s - a)|^2 ds = \int_{|B_s - a| \leq \epsilon} \left| \frac{1}{\epsilon} (B_s - a) - \text{sgn}(B_s - a) \right|^2 ds \rightarrow 0 a.s.$$

□

Theorem 15.5.

$$\mathcal{L}(t, a) = \int_0^t \delta_a(B_s) ds$$

Theorem 15.6. *If f is integrable on \mathbb{R} , then*

$$\int_{-\infty}^{\infty} \mathcal{L}(t, s) f(s) ds = \int_0^t f(B_s) ds$$

16. LECTURE 16 - THURSDAY 21 MARCH

Definition 16.1 (Linear cointegration). Consider two non stationary time series X_t and Y_t . If there exist coefficients α and β such that

$$\alpha X_t + \beta Y_t = u_t$$

with u_t stationary, then we say that X_t and Y_t are **cointegrated**.

Definition 16.2 (Nonlinear cointegration). If $Y_t - f(X_t) = u_t$ is stationary, with $f(\cdot)$ a nonlinear function.

17. LECTURE 17 - TUESDAY 3 MAY

17.1. Stochastic integrals for martingales. We now seek to define stochastic integrals with respect to processes other than Brownian motion.

Example 17.1. Let $X_t = \int_0^t Y_s dB_s$, and thus $dX_s = Y_s dB_s$. Then

$$Z_t = \int_0^t Y_s' dX_s = \int_0^t Y_s' \cdot Y_s dB_s$$

Revus and Yov - Continuous Martingale and Brownian Motion

Definition 17.2 (Martingale). A martingale with respect

- (1) M_t adapted to \mathcal{F}_t .
- (2) $\mathbb{E}(|M_t|) < \infty$.

(3) $\mathbb{E}(M_t | \mathcal{F}_s) = M_s$ a.s.

A process is a submartingale if (3) is replaced with $\mathbb{E}(M_t | \mathcal{F}_s) \geq M_s$. A process is a supermartingale if (3) is replaced with $\mathbb{E}(M_t | \mathcal{F}_s) \leq M_s$.

Example 17.3. Let N_t be a poisson process with intensity λ . Then

- (1) N_t is a submartingale with respect to the natural filtration.
- (2) $N_t - \lambda t$ is a martingale with respect to the natural filtration.

Theorem 17.4. If $\int_0^t \mathbb{E}(H^2(s)) ds < \infty$ and $H(s)$ is adapted to $\mathcal{F}_s = \sigma(B_t, t \leq s)$, then

$$Y_t = \int_0^t H(s) dB_s, t \geq 0$$

is a continuous, square integrable martingale - that is, $\mathbb{E}(Y_t^2) < \infty$.

Theorem 17.5. Let M_t be a continuous, square integrable martingale with respect to \mathcal{F}_t . Then there exists an adapted process $H(s)$ such that $\int_0^t \mathbb{E}(H^2(s)) ds < \infty$ and

$$M_t = M_0 + \int_0^t H(s) dB_s$$

where B_t is a Brownian motion with respect to \mathcal{F}_t .

Theorem 17.6. $M_t, t \geq 0$ is a Brownian motion if and only if it is a local continuous martingale with $[M, M](t) = t, t \geq 0$ under some probability measure Q .

Proof. A local continuous martingale is of the form $M_t = M_0 + \int_0^t H(s) dB_s$. Then we have

$$[M, M](t) = \int_0^t H^2(s) ds = t \Rightarrow H(s) = 1 \text{ a.s.} \Rightarrow M_t = B_t.$$

□

Theorem 17.7. Let $M_t, t \geq 0$ be a continuous local martingale such that $[M, M](t) \uparrow \infty$. Let

$$\tau_t = \inf\{s : [M, M](s) \geq t\}.$$

Then $M(\tau_t)$ is a Brownian motion. Moreover, $M(t) = B([M, M](t)), t \geq 0$.

This is an application of the **change of time** method.

Example 17.8. B_t is a Brownian motion - and then $Y_t = B_t^2 - t$ is a martingale. We have

$$dY_t = H(s) dB_s = 2B_s dB_s.$$

Thus,

$$B_t^2 - t = 2 \int_0^t B_s dB_s.$$

Definition 17.9 (Predictability of a stochastic process). A stochastic process $X_t, t \geq 0$ is said to be predictable with respect to \mathcal{F}_t if $X_t \in \mathcal{F}_{t-}$ for all $t \geq 0$, where

$$\mathcal{F}_{t-} = \bigcap_{h \geq 0} \mathcal{F}_{t+h}, \quad \mathcal{F}_{t-} = \sigma \left(\bigcap_{h > 0} \mathcal{F}_{t-h} \right).$$

Example 17.10. Let g_t be a step process, with

$$g_t = \sum_{i=1}^n \zeta_i \mathbf{1}_{[t_i, t_{i+1})}(t)$$

Then g_t is not predictable.

Let g_t be a step process, with

$$g_t = \sum_{i=1}^n \zeta_i \mathbf{1}_{(t_i, t_{i+1}]}(t)$$

Then g_t is predictable.

Example 17.11. Let N_t be a Poisson process. Then N_{t-} is predictable, but N_t is not predictable.

From now on, assume $M_t, t \geq 0$ is right continuous, square integrable martingale with left hand limits.

Lemma 17.12. M_t^2 is a submartingale.

Proof.

$$\begin{aligned} \mathbb{E}(M_t^2 | \mathcal{F}_s) &= \mathbb{E}(M_s^2 + 2(M_t - M_s) + (M_t - M_s)^2 | \mathcal{F}_s) \\ &= M_s^2 + \mathbb{E}((M_t - M_s)^2 | \mathcal{F}_s) \\ &\geq M_s^2 \end{aligned}$$

□

Theorem 17.13 (Doob-Myer decomposition). *By Doob-Myer we can write*

$$M_t^2 = L_t + A_t$$

where L_t is a martingale, and A_t is a predictable process, right continuous, and increasing, such that $A_0 = 0, \mathbb{E}(A_t) < \infty, t \geq 0$.

A_t is called the **compensator** of M_t^2 , and is denoted by $\langle M, M \rangle(t)$.

Example 17.14. Consider $B_t^2 = B_t^2 - t + t$. Then

$$\langle B, B \rangle(t) = t = [B, B](t)$$

Theorem 17.15. *If M_t is continuous then*

$$[M, M](t) = \langle M, M \rangle(t).$$

Example 17.16. Let N_t be a Poisson process. Then we know $\tilde{N}_t = N_t - \lambda t$ is a martingale. We may prove

$$\tilde{N}_t^2 - \lambda t$$

is a martingale, that is, $\langle \tilde{N}, \tilde{N} \rangle(t) = \lambda t$. However,

$$[\tilde{N}, \tilde{N}](t) = \lambda t + \tilde{N}_t \neq \langle \tilde{N}, \tilde{N} \rangle(t)$$

Example 17.17. $X_t = \int_0^t f(s) dB_s$ is a continuous martingale. Thus,

$$[X, X](t) = \int_0^t f^2(s) ds = \langle X, X \rangle(t)$$

Theorem 17.18. If M_t is a continuous, square integrable martingale, then

$$M_t^2 - [M, M](t)$$

is a martingale, and so

$$[M, M](t) = \langle M, M \rangle(t) + \text{martingale}$$

which implies

$$\mathbb{E}[M, M](t) = \mathbb{E}\langle M, M \rangle(t) = \mathbb{E}M_t^2$$

We now turn to defining integrals such as

$$\int_0^t X_s dM_s$$

where M_s is a martingale.

Definition 17.19. Let L_{pred}^2 be the space of all predictable stochastic process X_s satisfying the condition

$$\int_0^t X_s^2 d\langle M, M \rangle(s) < \infty.$$

Then the integral $\int_0^t X_s dM_s$ is defined as before in two steps.

(1) If $X_s \in L_{pred}^2$ and $X_s = \sum_{j=1}^n \zeta_j \mathbf{1}_{[t_j, t_{j+1})}$. Define

$$I(X) = \sum_{j=1}^n \zeta_j (M_{t_{j+1}} - M_{t_j}).$$

(2) For all $X_s \in L_{pred}^2$, there exists a sequence of step process X_s^n such that $X_s^n \rightarrow X_s$ in L^2 . Define $I(x)$ to be the limit in such situations such that

$$\mathbb{E}(I(X) - I(X^n))^2 \rightarrow 0.$$

Proposition 17.20. *Properties of the integral.*

(1) If M_s is a (local) martingale, then

$$\int_0^t f(s) dM_s$$

is a (local) martingale.

Proof.

$$\mathbb{E}\left(\int_s^t f(u) dM_u \mid \mathcal{F}_s\right) = 0$$

□

(2) If M_t is a square integrable martingale and satisfies

$$\mathbb{E}\left(\int_0^t f^2(s) d\langle M, M \rangle(s)\right) < \infty$$

then

$$I(f) = \int_0^t f(s) dM_s$$

is square integrable with $E(I(f)) = 0$, and $E(I^2(f)) = \int_0^t f^2(s) d\langle M, M \rangle(s)$.

In particular, if $M(s) = \int_0^s \sigma(u) dB_u$, then

$$\int_0^t X_s dM_s = \int_0^t X_s \sigma(s) dB_s$$

provided $\int_0^t X_s^2 \sigma^2(s) ds < \infty$ and $\int_0^t \sigma^2(s) ds$

(3) If $X_t = \int_0^t f(x) dM_s$ and M_s is a continuous, square integrable martingale, then

$$[X, X](t) = \int_0^t f^2(s) d[M, M](s) = \langle X, X \rangle(t)$$

18. LECTURE 18 - TUDSAY 10 MAY

Let M_t be a martingale. Then $M_t^2 - [M, M](t)$ is a martingale, and

$$M_t^2 = \text{martingale} + \langle M, M \rangle(t)$$

Recall that if M_t is a continuous square integrable martingale, then

$$\langle M, M \rangle(t) = [M, M](t)$$

Generally speaking,

$$[M, M](t) = \langle M, M \rangle(t) + \text{martingale}$$

Theorem 18.1. *If*

$$\int_0^t f^2(s) d\langle M, M \rangle(s) < \infty$$

a.s. then

$$Y_t = \int_0^t f(s) dM_s$$

is well defined, and

$$\begin{aligned} \mathbb{E}Y_t^2 &= \int_0^t f^2(s) d\langle M, M \rangle(t) \\ &= \int_0^t f^2(s) d[M, M](t) \quad \text{if } M_t\text{-continuous} \end{aligned}$$

and

$$\langle Y, Y \rangle(t) = [Y, Y](t) = \int_0^t f^2(s) d[M, M](s).$$

Proof.

$$\begin{aligned} dY_t &= f(t) dM_t \\ d[Y, Y](t) &= dY_t dY_t \\ &= f^2(t) dM_t dM_t \\ &= f^2(t) d[M, M](t) \\ [Y, Y](t) &= \int_0^t f^2(s) d[M, M](s) = \langle Y, Y \rangle(t) \end{aligned}$$

By Itô's formula, we also have

$$\begin{aligned} Y_t^2 &= Y_0^2 + 2 \int_0^t Y_s dY_s + \int_0^t 1 \cdot [Y, Y](t) \\ &= 2 \int_0^t Y_s \cdot f(s) dM_s + \langle Y, Y \rangle(t) \\ dY_t^2 &= 2Y_t f(t) dM_t + d\langle Y, Y \rangle(t) \end{aligned}$$

Hence

$$\langle Y, Y \rangle(t) = Y_t^2 - 2 \int_0^t Y_s f(s) dM_s = [Y, Y](t)$$

since $\int_0^t Y_s f(s) dM_s$ is a martingale. □

18.1. Itô's integration for continuous semimartingales.

Definition 18.2. Let (X_t, \mathcal{F}_t) be a continuous semimartingale. Then

$$X_t = M_t + A_t$$

where M_t is a martingale and A_t is a continuous adapted process of bounded variation ($\lim_{\delta \rightarrow 0} \sum_{j=1}^n |A_{t_{j+1}} - A_{t_j}| < \infty$).

Definition 18.3 (Integrals for semimartingales).

$$\int_0^t c_s dX_s = \int_0^t c_s dM_s + \int_0^t c_s dA_s$$

and since A_t has bounded variation, the second integral is defined in the Riemann-Stiljes (R.S.) sense.

Theorem 18.4 (Itô's formula). *Let X_t be a continuous semimartingale. Let $f(x)$ have twice continuous derivatives. Then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d[X, X](s)$$

Proof. Partition the interval $[0, t]$, and use a Taylor expansion to express $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$. \square

18.2. Stochastic differential equations. Consider the equation

$$dX_t = \sigma(t, X_t) dB_t + \mu(t, X_t) dt$$

We seek to solve for a function $f(t, x)$ such that

$$X_t = f(t, B_t).$$

Such an $f(t, B_t)$ is a **solution** to the stochastic differential equation.

Definition 18.5 (Strong solution). $X_t = X_0 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t \mu(s, X_s) ds$

19. LECTURE 19 - THURSDAY 12 MAY

Theorem 19.1. *Let*

$$dX_t = a(t, X_t) dt + b(t, X_t) dB_t$$

Assume $\mathbb{E}X_0 < \infty$. X_0 is independent of B_s and there exists a constant $c > 0$ such that

- (1) $|a(t, x)| + |b(t, x)| \leq C(1 + |x|)$.
- (2) $a(t, x), b(t, x)$ satisfy the Lipschitz condition in x , i.e.

$$|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leq C|x - y|$$

for all $t \in (0, T)$.

*Then there exists a **unique (strong) solution**.*

Example 19.2. Let

$$dX_t = c_1 X_t dt + c_2 X_t dB_t,$$

with c_1, c_2 constants.

Example 19.3. Let

$$dX_t = [c_1(t)X_t + c_2(t)] dt + [\sigma_1(t)X_t + \sigma_2(t)] dB_t$$

Let $a(t, x) = c_1(t)x + c_2(t)$, $b(t, x) = \sigma_1(t)x + \sigma_2(t)$. Just follow Kuo p. 233.

Let

$$H_t = e^{-Y_t}, \quad Y_t = \int_0^t \sigma_1(s) ds + \int_0^t c_1(s) dB_s - \frac{1}{2} \int_0^t c_1^2(s) ds$$

Then by the Itô product formula, we have

$$d(H_t X_t) = H_t (dX_t - \sigma_1(t)X_t dt - c_1(t)X_t dB_t - c_2(t)c_1(t) dt)$$

Then by definition of the X_t , we obtain

$$d(H_t X_t) = H_t (c_2(t) dB_t + \sigma_2(t) dt - c_1(t)c_2(t) dt)$$

which can be integrated to yield

$$H_t X_t = C + \int_0^t H_s c_2(s) dB_s + \int_0^t H_s (\sigma_2(s) - c_1(s)c_2(s)) ds$$

Dividing both sides by H_t we obtain our solution X_t .

Theorem 19.4. *The solution to the linear stochastic differential equation*

$$dX_t = [c_1(t)X_t + c_2(t)] dt + [\sigma_1(t)X_t + \sigma_2(t)] dB_t$$

is given by

$$X_t = C e^{-Y_t} + \int_0^t e^{Y_t - Y_s} c_2(s) dB_s + \int_0^t e^{Y_t - Y_s} (\sigma_2(s) - c_1(s)c_2(s)) ds$$

where $Y_t = \int_0^t c_1(s) dB_s + \int_0^t (\sigma_1(s) - \frac{1}{2}c_1^2(s)) ds$

20. LECTURE 20 - TUESDAY 17 MAY

20.1. Numerical methods for stochastic differential equations.

Theorem 20.1 (Euler's method). *For the stochastic differential equation*

$$dX_t = a(X_t) dt + b(X_t) dB_t,$$

we simulate X_t according to

$$X_{t_j} = X_{t_{j-1}} + a(X_{t_{j-1}})\Delta t_j + b(X_{t_{j-1}})\Delta B_{t_j}$$

Theorem 20.2 (Milstein scheme). *For the stochastic differential equation*

$$dX_t = a(X_t) dt + b(X_t) dB_t,$$

we simulate X_t according to

$$X_{t_j} = X_{t_{j-1}} + a(X_{t_{j-1}})\Delta t_j + b(X_{t_{j-1}})\Delta B_{t_j} + \frac{1}{2}b'(X_{t_{j-1}})(\Delta B_{t_j}^2 - \Delta t_j)$$

20.2. Applications to mathematical finance.

20.3. Martingale method. Consider a market with risky security S_t and riskless security β_t .

Definition 20.3 (Contingent claim). A random variable $C_T : \Omega \rightarrow \mathbb{R}$, \mathcal{F}_T -measurable is called a contingent claim. If C_T is $\sigma(S_T)$ -measurable it is **path-independent**.

Definition 20.4 (Strategy). Let a_t represent number of units of S_t , and b_t represent number of units of β_t . If a_t, b_t are \mathcal{F}_t -adapted, then they are strategies in our market model. Our strategy value V_t at time t is

$$V_t = a_t X_t + b_t \beta_t$$

Definition 20.5 (Self-financing strategy). A strategy (a_t, b_t) is self financing if

$$dV_t = a_t dS_t + b_t d\beta_t$$

The intuition is that we make one investment at $t = 0$, and after that only rebalance between S_t and β_t .

Definition 20.6 (Admissible strategy). (a_t, b_t) is an **admissible strategy** if it is self financing and $V_t \geq 0$ for all $0 \leq t \leq T$.

Definition 20.7 (Arbitrage). An arbitrage is an admissible strategy such that $V_0 = 0, V_T \geq 0$ and $\mathbb{P}(V_T > 0) > 0$. Alternatively, an arbitrage is a trading strategy with $V_0 = 0$, and $\mathbb{E}(V_T) > 0$.

Definition 20.8 (Attainable claim). A contingent claim C_T is said to be attainable if there exists an admissible strategy (a_t, b_t) such that $V_T = C_T$. In this case, the portfolio is said to replicate the claim. By the law of one price, $C_t = V_t$ at all t .

Definition 20.9 (Complete). The market is said to be **complete** if every contingent claim is attainable

Theorem 20.10 (Harrison and Pliska). Let \mathbb{P} denote the real world measure of the underlying asset price X_t . If the market is arbitrage free, there exists an equivalent measure \mathbb{P}^* , such that the discounted asset price \hat{X}_t and every discounted attainable claim \hat{C}_t are \mathbb{P}^* -martingales. Further, if the market is complete, then \mathbb{P}^* is unique. In mathematical terms,

$$C_t = \beta_t \mathbb{E}^*(\beta_T^{-1} C_T | \mathcal{F}_t).$$

\mathbb{P}^* is called the equivalent martingale measure (EMM) or the risk-neutral measure.

21. LECTURE 21 - THURSDAY 19 MAY

For a trading strategy (a_t, b_t) , then the value V_t satisfies

$$V_t = V_0 + \int_0^t \alpha_s dS_s + \int_0^t b_s d\beta_s$$

where B_s is the riskless asset.

To price an attainable option X , let (a_t, b_t) be a trading strategy with value V_t that replicates X . Then to avoid arbitrage, the value of X at time $t = 0$ is given by V_0 .

21.1. Change of Measure. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition 21.1 (Equivalent measure). Let \mathbb{P} and \mathbb{Q} be measures on (Ω, \mathcal{F}) . Then for any $A \in \mathcal{F}$, if

$$\mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0$$

then we say the measures \mathbb{P} and \mathbb{Q} are equivalent. If $\mathbb{P}(A) = 0 \Rightarrow \mathbb{Q}(A) = 0$, we write $\mathbb{Q} \ll \mathbb{P}$.

Theorem 21.2 (Radon-Nikodym). *Let $\mathbb{Q} \ll \mathbb{P}$. Then there exists a random variable λ such that $\lambda \geq 0$, $\mathbb{E}_{\mathbb{P}}(\lambda) = 1$ and*

$$\mathbb{Q}(A) = \int_A d\mathbb{P} = \mathbb{E}_{\mathbb{P}}(\lambda \mathbf{1}_A)$$

for any $A \in \mathcal{F}$. λ is \mathbb{P} -almost surely unique.

Conversely, if there exists λ such that $\lambda \geq 1$, $\mathbb{E}_{\mathbb{P}}(\lambda) = 1$, then defining

$$\mathbb{Q}(A) = \int_A \lambda d\mathbb{P}$$

and then \mathbb{Q} is a probability measure and $\mathbb{Q} \ll \mathbb{P}$. Consequently, if $\mathbb{Q} \ll \mathbb{P}$, then

$$\mathbb{E}_{\mathbb{Q}}(Z) = \mathbb{E}_{\mathbb{P}}(\lambda Z)$$

whenever $\mathbb{E}_{\mathbb{Q}}(|Z|) < \infty$.

The random variable λ is called the density of \mathbb{Q} with respect to \mathbb{P} , and denoted by

$$\lambda = \frac{d\mathbb{Q}}{d\mathbb{P}}$$

Example 21.3. Let $X \sim N(0, 1)$ and $Y \sim N(\mu, 1)$ under probability \mathbb{P} . Then there exists a \mathbb{Q} such that \mathbb{Q} is equivalent to \mathbb{P} and $Y \sim N(0, 1)$ under \mathbb{Q} .

Proof.

$$\begin{aligned}\mathbb{P}(X \in A) &= \frac{1}{\sqrt{2\pi}} \int_A e^{-\frac{t^2}{2}} dt \\ \text{Define } \mathbb{Q}(A) &= \int_A e^{-\mu X - \frac{\mu^2}{2}} d\mathbb{P} \\ &= \frac{1}{\sqrt{2\pi}} \int_A e^{-\mu x - \frac{\mu^2}{2}} e^{-\frac{x^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_A e^{-\frac{(\mu+x)^2}{2}} dx\end{aligned}$$

□

Then $\mathbb{Q} \ll \mathbb{P}, \mathbb{P} \ll \mathbb{Q}$ and let

$$\lambda = \frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\mu X - \frac{\mu^2}{2}}$$

Then λ satisfies the conditions of Radon-Nikodym theorem.

Then we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(Y) &= \mathbb{E}_{\mathbb{P}}((X + \mu)\lambda) \\ &= \int (X + \mu) e^{-\mu X - \frac{\mu^2}{2}} d\mathbb{P} \\ &= \frac{1}{\sqrt{2\pi}} \int (x + \mu) e^{-\frac{(\mu+x)^2}{2}} dx = 0\end{aligned}$$

22. LECTURE 22 - TUESDAY 24 MAY

Theorem 22.1. Let $\lambda(t), 0 \leq t \leq T$ be a positive martingale with respect to \mathcal{F}_t such that

$$\mathbb{E}_{\mathbb{P}}(\lambda(T)) = 1.$$

Define a new probability measure \mathbb{Q} by

$$\mathbb{Q}(A) = \int_A \lambda(T) d\mathbb{P}$$

Then $\mathbb{Q} \ll \mathbb{P}$ and for any random variable X , we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(X) &= \mathbb{E}_{\mathbb{P}}(\lambda(T)X) \\ \mathbb{E}_{\mathbb{Q}}(X | \mathcal{F}_t) &= \mathbb{E}_{\mathbb{P}}\left(\frac{\lambda(T)X}{\lambda(t)} | \mathcal{F}_t\right) \\ &= \frac{\mathbb{E}_{\mathbb{P}}(\lambda(T)X | \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\lambda(T) | \mathcal{F}_t)} \text{ a.s.}\end{aligned} \tag{*}$$

and if $X \in \mathcal{F}_t$, then for any $s \leq t$, we have

$$\mathbb{E}_{\mathbb{Q}}(X | \mathcal{F}_s) = \mathbb{E}_{\mathbb{P}}\left(\frac{\lambda(t)X}{\lambda(s)} | \mathcal{F}_s\right) \tag{**}$$

Consequently a process $S(t)$ is a \mathbb{Q} -martingale if and only if

$$S(t)\lambda(t) \tag{†}$$

is a \mathbb{P} -martingale

Proof. (★). We have

$$\begin{aligned} \mathbb{Q}(\mathbb{E}_{\mathbb{P}}(\lambda(T)) | \mathcal{F}_t = 0) &= \mathbb{E}_{\mathbb{P}}(\lambda(T)\mathbf{1}_{\mathbb{E}_{\mathbb{P}}(\lambda(T) | \mathcal{F}_t=0)}) \\ &= 0 \end{aligned}$$

We have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left(\frac{\mathbb{E}_{\mathbb{P}}(\lambda(T)X | \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\lambda(T) | \mathcal{F}_t)} \mathbf{1}_A \right) &= \mathbb{E}_{\mathbb{Q}} \left(\lambda(t) \frac{\mathbb{E}_{\mathbb{P}}(\lambda(T)X | \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\lambda(T) | \mathcal{F}_t)} \mathbf{1}_A \right) \\ &= \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\lambda(T)X | \mathcal{F}_t) \mathbf{1}_A) \\ &= \mathbb{E}_{\mathbb{P}}(\lambda(T)X \mathbf{1}_A) \\ &= \mathbb{E}_{\mathbb{Q}}(X \mathbf{1}_A) \end{aligned}$$

(★★). We have

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left(\frac{\lambda(t)X}{\lambda(s)} | \mathcal{F}_s \right) &= \frac{1}{\lambda(s)} \mathbb{E}_{\mathbb{P}}(\lambda(t)X | \mathcal{F}_s) \\ &= \frac{1}{\lambda(s)} \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\lambda(T)X | \mathcal{F}_t) | \mathcal{F}_s) \\ &= \frac{1}{\lambda(s)} \mathbb{E}_{\mathbb{P}}(\lambda(T)X | \mathcal{F}_s) \\ &= \mathbb{E}_{\mathbb{Q}}(X | \mathcal{F}_s) \end{aligned}$$

because of (★).

(†). We have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(S(t) | \mathcal{F}_u) &= S(u) \\ \iff \mathbb{E}_{\mathbb{P}} \left(\frac{\lambda(t)S(t)}{\lambda(u)} | \mathcal{F}_u \right) &= S(u) \\ \iff \mathbb{E}_{\mathbb{P}}(\lambda(t)S(t) | \mathcal{F}_u) &= \lambda(u)S(u) \end{aligned}$$

as required. □

Theorem 22.2. Let $B_s, 0 \leq s \leq T$ be a Brownian motion under \mathbb{P} . Let $S(t) = B_t + \mu t, u \neq 0$. Then there exists a \mathbb{Q} equivalent to \mathbb{P} such that $S(t)$ is a \mathbb{Q} -Brownian motion and

$$\lambda(T) = \frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\mu B_T - \frac{1}{2}\mu^2 T}.$$

Note that $S(t)$ is not a martingale under \mathbb{P} , but it is a martingale under \mathbb{Q} .

Proof. Under \mathbb{Q} ,

$$\mathbb{Q}(B_0 = 0) = \int_{B_0=0} \lambda(T) d\mathbb{P} = 1$$

$S(t)$ is a \mathbb{Q} -martingale if and only if $S(t)\lambda(t)$ is a \mathbb{P} -martingale. But we have

$$X_t = S(t)\lambda(t) = (B_t + \mu t) e^{-\mu B_t - \frac{1}{2}\mu^2 t}$$

is a martingale.

Finally, note that

$$[S, S](t) = [B, B](t) = t$$

□

22.1. Black-Scholes model.

Definition 22.3 (Black-Scholes model). The Black-Scholes model assumes the risky asset S_t follows the diffusion process given by

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

and the riskless asset follows the diffusion

$$\frac{d\beta_t}{\beta_t} = r dt$$

Define the discounted process as follows:

$$\hat{S}_t = \frac{S_t}{\beta_t}, \quad \hat{V}_t = \frac{V_t}{\beta_t}, \quad \hat{C}_t = \frac{C_t}{\beta_t}.$$

23. LECTURE 23 - THURSDAY 26 MAY

Lemma 23.1.

(a) By a simple application of Itô's lemma,

$$\frac{d\hat{S}_t}{\hat{S}_t} = (\mu - r) dt + \sigma dB_t.$$

(b) \hat{S}_t is a \mathbb{Q} -martingale with

$$\lambda = \frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-qB_T - \frac{1}{2}q^2 T}$$

with $q = \frac{\mu - r}{\sigma}$.

(c) Note that

$$\frac{d\hat{S}_t}{\hat{S}_t} = \sigma d(B_t + \frac{\mu - r}{\sigma} t) = \sigma d\hat{B}_t$$

where $\hat{B}_t = B_t + qt$ is a Brownian motion under \mathbb{Q} .

(d) $d\hat{S}_t = \sigma \hat{S}_t d\hat{B}_t$.

(e) In a finite market, where S_t takes only finitely many values, \hat{S}_t is a \mathbb{Q} -martingale is a necessary condition for no-arbitrage.

(f) Note that

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t = r dt + \sigma d\hat{B}_t$$

Theorem 23.2. A value process V_t is self financing if and only if the discounted value process \hat{V}_t is a \mathbb{Q} -martingale.

$$\begin{aligned} d\hat{V}_t &= a_t d\hat{S}_t && \iff dV_t = \alpha dS_t + b_t d\beta_t \\ \iff V_t &= V_0 + \int_0^t a_s dS_s + \int_0^t b_s d\beta_s \\ \iff V_t &\text{ is self financing.} \end{aligned}$$

Proof. By Itô's formula, we have

$$\begin{aligned} d\hat{V}_t &= e^{-rt} dV_t - re^{-rt} V_t dt \\ &= e^{-rt} (a_t dS_t + b_t d\beta_t) - re^{-rt} (a_t S_t + b_t \beta_t) dt \\ &= a_t (e^{-rt} dS_t - re^{-rt} dt) \\ &= a_t d\hat{S}_t \end{aligned}$$

□

Theorem 23.3. In the Black-Scholes model, there are no arbitrage opportunities.

Proof. For any admissible trading strategy (a_t, b_t) , we have that the discounted value process \hat{V}_t is a \mathbb{Q} -martingale. So if $V_0 = 0$, then $\mathbb{E}(\hat{V}_0) = 0$, and we have

$$\mathbb{E}_{\mathbb{Q}}(\hat{V}_T) = \mathbb{E}_{\mathbb{Q}}(\hat{V}_T | \mathcal{F}_0) = \hat{V}_0 = 0$$

which implies $\mathbb{Q}(\hat{V}_T > 0) = 0$, which implies $\mathbb{P}(V_T > 0) = 0$, which implies that $\mathbb{E}_{\mathbb{P}}(V_T) = 0$, which then implies no arbitrage. □

Theorem 23.4. For any self financing strategy,

$$V_t = a_t S_t + b_t \beta_t = V_0 + \int_0^t a_u dS_u + \int_0^t b_u d\beta_u$$

And so a strategy is self financing if

$$S_t da_t + \beta_t db_t + d[a, S](t) = 0$$

We now consider several cases.

(1) If a_t is of bounded variation, then $[a, S](t) = 0$. Hence

$$S_t da_t + \beta_t db_t = 0,$$

which implies

$$db_t = -\frac{S_t}{\beta_t} da_t$$

Hence $da_t \cdot db_t < 0$.

- (2) If a_t is a semi-martingale $a_t = a_t^2 + A_t$, then b_t must be a semi-martingale, where $b_t = b_t^2 + B_t$ where a_t^2, b_t^2 are the martingale parts and A_t, B_t are of bounded variation.

24. LECTURE 24 - TUESDAY 31 MAY

Theorem 24.1. Given a claim C_T under the self-financing assumption, there exists a \mathbb{Q} -martingale such that

$$V_t = \mathbb{E}_{\mathbb{Q}} \left(e^{-r(T-t)C_T} \mid \mathcal{F}_t \right), \mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t).$$

In particular, we have

$$V_0 = \mathbb{E}_{\mathbb{Q}}(e^{-rT}C_T)$$

Proof. For any \mathbb{Q} -martingale \hat{V}_t , we have $\hat{V}_t = \mathbb{E}_{\mathbb{Q}}(\hat{V}_T \mid \mathcal{F}_t)$. Then

$$V_t = \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)C_T} \mid \mathcal{F}_t),$$

as $V_T = C_T$. □

Theorem 24.2. A claim is attainable (there exists a trading strategy replicating the claim), that is,

$$\begin{aligned} V_t &= V_0 + G(t) \\ G(t) &= \int_0^t a_u dS_u + \int_0^t b_u d\beta_u \\ V_T &\geq 0, V_T = C_T \end{aligned}$$

Theorem 24.3. Suppose that C_T is a non-negative random variable, $C_t \in \mathcal{F}_T$ and $\mathbb{E}_{\mathbb{Q}}(C_T^2) < \infty$, where \mathbb{Q} is defined as before. Then

(a) The claim is replicable.

(b)

$$V_t = \mathbb{E}_{\mathbb{Q}} \left(e^{-r(T-t)C_T} \mid \mathcal{F}_t \right) \iff \hat{V}_t = \mathbb{E}_{\mathbb{Q}}(\hat{C}_T \mid \mathcal{F}_t)$$

where $\hat{C}_T = e^{-rT}C_T$.

In particular, $V_0 = \mathbb{E}_{\mathbb{Q}}(e^{-rT}C_T) = \mathbb{E}_{\mathbb{Q}}(\hat{C}_T)$.

Theorem 24.4. Assume $\hat{V}_t = \mathbb{E}_{\mathbb{Q}}(\hat{C}_T \mid \mathcal{F}_t)$. Using the martingale representation theorem, there exists an adapted process $H(s)$ such that

$$\hat{V}_t = \hat{V}_0 + \int_0^t H(s) d\hat{B}_s \iff d\hat{V}_t = H(t) d\hat{B}_t.$$

On the other hand, $d\hat{V}_t = a_t d\hat{S}_t = a_t \cdot \sigma \hat{S}_t d\hat{B}_t$. Hence we obtain our required result,

$$a_t = \frac{H(t)}{\sigma \hat{S}_t},$$

and then solve for b_t .

Example 24.5. Let $C_T = f(S_T)$. Then $V_t = \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}f(S_T)|\mathcal{F}_t)$. Since $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t) = \sigma(\hat{B}_s, 0 \leq s \leq t)$, and \hat{S}_t is a \mathbb{Q} -martingale, we have

$$\hat{S}_t = \hat{S}_0 e^{-\frac{\sigma^2}{2}t + \sigma \hat{B}_t}$$

and so

$$S_T = e^{rT} \hat{S}_T = \hat{S}_t e^{rT} e^{-\frac{\sigma^2}{2}(T-t) + \sigma(\hat{B}_T - \hat{B}_t)}.$$

Then

$$V_t = \mathbb{E}_{\mathbb{Q}} \left[e^{-r(T-t)} f \left(e^{rT} \hat{S}_t e^{-\frac{\sigma^2}{2}(T-t) + \sigma(\hat{B}_T - \hat{B}_t)} \right) \right]$$

and so

$$V_t = F(t, S_t)$$

where

$$\begin{aligned} F(t, x) &= \mathbb{E}_{\mathbb{Q}} \left[e^{-r(T-t)} f \left(e^{(-\frac{\sigma^2}{2})(T-t) + \sigma\sqrt{T-t}Z} \right) \right] \\ &= e^{-r(T-t)} \int_{\mathbb{R}} f \left(x e^{-\frac{\sigma^2}{2}(T-t) + \sigma z\sqrt{T-t}} \right) \phi(z) dz \end{aligned}$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$.

Example 24.6. In particular, if $f(y) = (y - K)^+$, then we obtain

$$\begin{aligned} F(t, x) &= e^{-r\theta} \int_{-d'_1}^{\infty} x e^{-\frac{\sigma^2}{2}\theta + xz\sqrt{\theta} - \frac{z^2}{2}} dz - K \int_{-d'_1}^{\infty} e^{-r\theta - \frac{z^2}{2}} dz \\ &= x\Phi(d'_1 + \sigma\sqrt{\theta}) - Ke^{-r\theta}\Phi(d'_1) \\ &= x\Phi(d_1) - Ke^{-r\theta}\Phi(d_2), \end{aligned}$$

where

$$d_1 = \frac{\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}}, \quad d_2 = d_1 - \sigma\sqrt{\theta}$$

Theorem 24.7 (Black-Scholes model summary). $V_t = a_t S_t + b_t \beta_t$, where

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sigma dB_t \\ \frac{d\beta_t}{\beta_t} &= r dt \end{aligned}$$

(a) $\hat{S}_t = e^{-rt} S_t$ is a \mathbb{Q} -martingale, where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-qB_T - \frac{1}{2}q^2T}$$

and $q = \frac{\mu-r}{\sigma}$.

Then $\hat{B}_t = B_t + \frac{\mu-r}{\sigma}t$ is a \mathbb{Q} -Brownian motion, and $d\hat{S}_t = \sigma\hat{S}_t d\hat{B}_t$.

(b) V_t is self-financing if (a_t, b_t) satisfies

$$S_t da_t + \beta_t db_t + d[a, S](t) = 0$$

which then implies \hat{V}_t is a \mathbb{Q} -martingale, $\hat{V}_t = e^{-rt} V_t$.

(c) There are no arbitrage opportunities in the Black-Scholes model.

25. LECTURE 25 - THURSDAY 2 JUNE

Theorem 25.1 (Feynman-Kac formula).

(1) Suppose the function $F(x, t)$ solves the boundary value problem

$$\frac{\partial F(t, x)}{\partial t} + \mu(t, x) \frac{\partial F(t, x)}{\partial x} + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 F(t, x)}{\partial x^2} = 0$$

such that $F(T, x) = \Psi(x)$.

(2) Let S_t be a solution of the SDE

$$dS_t = \mu(t, S_t) dt + \sigma(t, S_t) dB_t \quad (\star)$$

where B_t is a \mathbb{Q} -Brownian motion

(3) Assume

$$\int_0^T \mathbb{E}(\sigma(t, S_t) \frac{\partial^2 F(t, S_t)}{\partial x^2}) dt < \infty$$

Then

$$F(t, S_t) = \mathbb{E}_{\mathbb{Q}}(\Psi(S_T) | \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(F(T, S_T) | \mathcal{F}_t).$$

where $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$.

Proof. It is enough to show that $F(t, S_t)$ is a martingale with respect to \mathcal{F}_t under \mathbb{Q} . By Itô's lemma, we have

$$\begin{aligned} dF(t, S_t) &= \frac{\partial f(t, S_t)}{\partial t} dt + \frac{\partial F(t, S_t)}{\partial x} dS_t + \frac{1}{2} \frac{\partial^2 F(t, S_t)}{\partial x^2} \cdot (dS_t)^2 \\ &= \left[\frac{\partial F}{\partial t} + \mu(t, S_t) \frac{\partial F}{\partial x} + \frac{\sigma^2(t, S_t)}{2} \frac{\partial^2 F(t, S_t)}{\partial x^2} \right] dt + \frac{\partial F(t, S_t)}{\partial x} \sigma(t, S_t) dB_t \\ &= \frac{\partial F(t, S_t)}{\partial x} \sigma(t, S_t) dB_t \end{aligned}$$

which is a \mathbb{Q} -martingale. □

Theorem 25.2 (General Feynman-Kac formula). *Let S_t be a solution of the SDE (\star) . Assume that there is a solution to the PDE*

$$\frac{\partial F(t, x)}{\partial t} + \mu(t, x) \frac{\partial F(t, x)}{\partial x} + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 F(t, x)}{\partial x^2} = r(t, x) F(t, x).$$

Then

$$F(t, S_t) = \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_t^T r(u, S_u) du} F(T, S_T) \mid \mathcal{F}_t \right)$$

Proof. Again by Itô's lemma,

$$\begin{aligned} dF(t, S_t) &= \left(\frac{\partial F}{\partial t} + \mu \frac{\partial F}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 F}{\partial x^2} \right) dt + \frac{\partial F}{\partial x} \cdot \sigma(t, S_t) dB_t \\ &= r(t, x) F(t, S_t) dt + dM_t \end{aligned}$$

where $M_t = \int_0^t \frac{\partial F}{\partial x} \sigma(u, S_u) dB_u$. Hence we have

$$dF(t, S_t) = r(t, S_t) X_t dt + dM_t$$

$$d \left[e^{-\int_t^\nu r(u, S_u) du} X_\nu \right] = e^{-\int_t^\nu r(u, S_u) du} dM_\nu \quad (\Rightarrow)$$

$$e^{-\int_t^T r(u, S_u) du} F(T, S_T) = F(t, S_t) + \int_t^T e^{-\int_t^\nu r(u, S_u) du} dM_\nu \quad (\Rightarrow)$$

$$\mathbb{E}_{\mathbb{Q}} \left(e^{-\int_t^T r(u, S_u) du} F(T, S_T) \mid \mathcal{F}_t \right) = F(t, S_t) \quad (\Rightarrow)$$

$$+ \underbrace{\mathbb{E}_{\mathbb{Q}} \left(\int_t^T e^{-\int_t^\nu r(u, S_u) du} \frac{\partial F}{\partial x} \sigma(\nu, S_\nu) dB_\nu \mid \mathcal{F}_t \right)}_{=0}$$

and so we obtain our result,

$$F(t, S_t) = \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_t^T r(u, S_u) du} F(T, S_T) \mid \mathcal{F}_t \right)$$

□