MATH 3964 - COMPLEX ANALYSIS

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1. Contour Integration and Cauchy’s Theorem

1.1. Analytic functions.

**Definition 1.1.** A function $f(z)$ is **differentiable** at $z_0$ if the limit

$$f'(z_0) = \lim_{\zeta \to z_0} \frac{f(\zeta) - f(z_0)}{\zeta - z_0}$$

exists independently of the path of approach.

**Definition 1.2.** A function $f(z)$ is **analytic** on a region $D$ if it is differentiable everywhere on $D$. Thus the derivative $f'(z)$ is a function defined on $D$. A function is analytic at a particular point $z_0 \in \mathbb{C}$ if it is differentiable on some open neighbourhood of $z_0$.

**Theorem 1.3.** A necessary condition for $f(z)$ to be analytic is that if $f(z) = u + iv$, then

$$v_y = u_x, v_x = -u_y$$

**Definition 1.4.** A function that is analytic throughout the whole complex plain is called an entire function.

**Definition 1.5.** A point at which a locally analytic function $f(z)$ fails to be analytic is a singularity of $f(z)$.

The easiest way to construct analytic functions is as sums of convergent power series. Any power series in the complex domain

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

having positive or infinite radius of convergence $R$ converges to an analytic function in the interior of its disc of convergence, $|z - z_0| < R$.

**Theorem 1.6.** Every convergent power series is differentiable term by term in the interior of its disc.

**Proposition 1.7.** The $n^{th}$ derivative of $f(z)$ at $z = z_0$ is

$$f^{(n)}(z_0) = n!a_n$$

The radius of convergence is given by either of the equivalent exact formulae:

$$R = \lim_{n \to \infty} \inf |a_n|^{-\frac{1}{n}}$$

or

$$\frac{1}{R} = \lim_{n \to \infty} \sup |a_n|^{\frac{1}{n}}$$

**Theorem 1.8.** Every power series with a positive or infinite radius of convergence is differentiable term by term to all orders in the interior of its disc of convergence.
1.2. **Contour integration.** A contour in the complex-plane is just a curve, finite or infinite, which has an arrow or orientation. We wish to assign a meaning to the contour integral,

$$\int_C f(z) \, dz$$

where $C$ is a contour and $f(z)$ is a function which is defined and piecewise continuous along $C$.

**Lemma 1.9** (Triangle inequality for contour integrals).

$$\left| \int_C f(z) \, dz \right| \leq \int_C |f(z)| \, |dz|$$

**Lemma 1.10** (The ML formula). If a contour $C$ has length $L$ and if $|f(z)| \leq M$ on $C$, then

$$\left| \int_C f(z) \, dz \right| \leq ML$$

**Lemma 1.11** (Jordan’s lemma). Let $C_R$ be all or part of the semicircular contour $Re^{i\theta}$, where $\theta$ runs from 0 to $\pi$. Suppose that $|f(z)| \leq M(R)$ on $C_R$ and $\lambda$ is a positive real number. Let

$$I(R) = \int_{C_R} f(z)e^{i\lambda z} \, dz$$

Then, we have the bound $|I(R)| = O(M(R))$.

1.3. **Cauchy’s theorem and extensions.**

**Theorem 1.12** (Cauchy’s theorem). If $f(z)$ is analytic on a simply connected region $D$ and if $C$ is any rectifiable closed contour or cycle in $D$, then

$$\int_C f(z) \, dz = 0.$$  

1.4. **Cauchy’s integral formula.**

**Theorem 1.13** (Cauchy’s integral formula). Suppose $f(z)$ is analytic in a simply connected region $D$ and that $C$ is a positively oriented rectifiable Jordan curve in $D$. Then

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \, dz = \begin{cases} f(z_0), & z_0 \in C \\ 0, & z_0 \notin C \end{cases}$$

**Theorem 1.14** (Analyticity of Cauchy integrals). The function

$$g(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

is differentiable to all orders in $D$ and is therefore analytic in $D$. It’s $n^{th}$ derivative is

$$g^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta.$$
Theorem 1.15. Suppose that $f(z)$ is analytic in a simply connected region $D$. Then
\[ f(z) = \frac{1}{2\pi i} \int_{C} \frac{f(\zeta)}{\zeta - z} d\zeta \]
and
\[ f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \]

Corollary. A function $f(z)$ has an antiderivative in a simply connected region $D$ if and only if $f(z)$ is analytic in $D$.

Theorem 1.16 (Removable singularities theorem). Suppose that $f(z)$ is analytic in a region $D$ except possibly at the point $z_1 \in D$. At $z_1$, suppose that
\[ \lim_{z \to z_1} (z - z_1)f(z) = 0 \]
Then a value of $f(z_1)$ can be assigned so that $f(z)$ becomes analytic at $z_1$.

Definition 1.17 (Analyticity at infinity). Suppose that $f(z)$ is analytic on an unbounded set and let $g(z) = f(1/z)$. Then the point $\infty \in \mathbb{C}^*$ is point of analyticity of $f(z)$ if $z = 0$ is a point of analyticity of $g(z)$. Similarly, $z = \infty$ is a singularity of a particular type of $f(z)$ if $g(z)$ has a singularity of that same type at $z = 0$.

Definition 1.18. A function $f(z)$ which has a singularity at $z_0$ and is analytic in a deleted neighbourhood of $z_0$ has a pole at $z_0$, or more specifically, a pole of order $k$, if $(z - z_0)^k f(z)$ is analytic and nonzero at $z_0$, where $k$ must be a positive integer according to the removable singularities theorem. A pole of order one is a simple pole, a pole of order two is a double pole, and so on.

Definition 1.19. A function is meromorphic if it is analytic in the whole complex plane $\mathbb{C}$ except for poles.

1.5. The Cauchy-Taylor theorem and analytic continuation.

Theorem 1.20 (Cauchy-Taylor theorem). Suppose that $f(z)$ is analytic at $z_0$ and the disc $D(R) = B(z_0, R)$ is the largest open disc on which $f(z)$ is analytic. Then the Taylor series,
\[ f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \]
converges absolutely to $f(z)$ on $D(R)$ and uniformly on compact subsets.

Theorem 1.21. Let $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ have a finite radius of convergence $R$. The radius of convergence of a power series is the distance from the centre to the nearest singularity of its sum function $f(z)$. 
Theorem 1.22 (Cauchy’s inequality). Let \( f(z) \) analytic on an open disc \( D(\rho) \) with centre \( z_0 \). Then, if \( |f(z)| \leq M(\rho) \), we have
\[
|f^{(n)}(z_0)| \leq \frac{n!M(\rho)}{\rho^n}
\]
If a power series \( \sum_{k=0}^{\infty} a_k(z - z_0)^k \) converges to \( f(z) \), then
\[
|a_n| \leq \frac{M(\rho)}{\rho^n}
\]

[Liouville’s theorem] If an entire function is bounded, or if it possibly grows at a rate such that \( f(z)/z \to 0 \) uniformly as \( z \to \infty \), then \( f(z) \) is constant.

Theorem 1.23 (Uniqueness of analytic continuation). Suppose that \( f(z), g(z) \) are analytic in a common region \( D \). Let \( H \) be a subset of \( D \) that contains a convergent subsequence \( \{z_k\} \) whose limit is in the interior of \( D \). If \( f(z) = g(z) \) for \( z \in H \), then \( f(z) = g(z) \) everywhere in \( D \).

1.6. Laurent’s theorem and the residue theorem.

Theorem 1.24 (Laurent’s theorem). Suppose that \( f(z) \) is analytic in the open circular annulus \( R_1 \leq |z - z_0| \leq R_2 \). Then \( f(z) \) admits a power series expansion in both positive and negative powers (a Laurent series),
\[
f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k
\]
which is absolutely convergent in \( D \) and uniformly convergent on compact subsets.

A formula for the coefficient \( a_n \) is
\[
a_n = \frac{1}{2\pi i} \int_{C(r)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta
\]

Definition 1.25. If \( z_0 \) is a pole or isolated essential singularity of \( f(z) \), then the residue of \( f(z) \) at \( z_0 \) is the coefficient \( a_{-1} \) of \( (z - z_0)^{-1} \) in the Laurent expansion of \( f(z) \) about \( z_0 \). The notation for the residue is \( \text{Res}(f, z_0) \).

Theorem 1.26 (Picard’s first theorem). A non-constant entire function has an image either the whole of the complex plane, with at most one exception.

Lemma 1.27. If \( f(z) \) has a simple pole at \( z_0 \), then
\[
\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0)f(z) = [(z - z_0)f(z)]_{z = z_0}
\]

Lemma 1.28. If \( f(z) \) has a pole of order \( k \) at \( z_0 \), then
\[
\text{Res}(f, z_0) = \frac{1}{(k-1)!} \left[ \frac{d^{k-1}}{dz^{k-1}} ((z - z_0)^k f(z)) \right]_{z = z_0}
\]
Lemma 1.29. If \( f(z) \) and \( h(z) \) are analytic at \( z_0 \) and \( h(z) \) has a simple zero at \( z_0 \), then
\[
\text{Res}(g/h, z_0) = \frac{g(z_0)}{h'(z_0)}
\]

Theorem 1.30 (Residue theorem). Suppose that \( f(z) \) is analytic inside and on a curve \( C \) except for a finite number of poles or isolated essential singularities \( z_i \) inside \( C \). Then
\[
\int_C f(z) \, dz = 2\pi i \sum_i \text{Res}(f, z_i)
\]

Definition 1.31 (Residue at infinity). Suppose that \( f(z) \) is analytic everywhere outside of a bounded region, in which it admits a convergent Laurent series. Then the residue of \( f(z) \) at infinity is minus the coefficient \( a_{-1} \), that is,
\[
\text{Res}(f, \infty) = -a_{-1}
\]

1.7. The Gamma function \( \Gamma(z) \).

Definition 1.32. For \( \Re(z) > 0 \), define
\[
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt
\]
The integral converges for \( \Re(z) > 0 \), and uniformly for \( \Re(z) \leq \delta > 0 \). Hence \( \Gamma(z) \) is analytic at least in the half-plane \( \Re(z) > 0 \).

Lemma 1.33 (Recurrence relation). Integration by parts gives \( \Gamma(z+1) = z\Gamma(z) \). This gives the analytic continuation to \( \Re(z) > -1 \). Repeated application provides the analytic continuation to the whole plane except for simple poles at \( z = 0, -1, -2, \ldots \).

Definition 1.34 (Beta function). For \( \Re(\alpha), \Re(\beta) > 0 \), define the function
\[
B(\alpha, \beta) = \int_0^1 t^\alpha (1-t)^{\beta-1} \, dt
\]

Theorem 1.35. We have the following relation
\[
B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}
\]

Lemma 1.36 (Duplication formula).
\[
\Gamma(\alpha + \frac{1}{2}) = \frac{\Gamma(2\alpha)\sqrt{\pi}}{2^{2\alpha-1}\Gamma(\alpha)}
\]

Lemma 1.37 (Functional relation).
\[
\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}
\]
1.8. The residue theorem.

Lemma 1.38. In all cases, including isolated essential singularities,
\[ \text{Res}(f, z_0) \]
is the coefficient of $\epsilon^{-1}$ in the Laurent expansion
\[ f(z_0 + \epsilon) = \sum_{n \in \mathbb{Z}} a_n \epsilon^n \]

Definition 1.39 (Residue at infinity). Suppose that $f(z)$ is analytic everywhere outside of a bounded region, in which it admits a convergent Laurent series. Then the residue of $f(z)$ at infinity is minus the coefficient $a_{-1}$, that is,
\[ \text{Res}(f, \infty) = -a_{-1} \]

If $f(z) \sim \frac{K}{|z|^p}$ as $|z| \to \infty$, then the residue at $\infty$ vanishes.

Theorem 1.40 (Argument principle). Suppose that $f(z)$ is analytic on and inside a positively oriented simple closed contour $C$. Suppose also that $f(z) \neq 0$ on $C$. Then
\[ \frac{1}{2\pi i} \oint_C \arg f(z) = N - P \]
where $N$ is the total number of zeroes and $P$ is the total number of poles inside $C$, counting multiplicities.

Example 1.41 (Integration of rational functions). Let $P(x)$ and $Q(x)$ be polynomials with $\deg P(x) = \deg Q(x) - 1$. Then
\[ \int_{\mathbb{R}} \frac{P(x)}{Q(x)} \, dx \]
does not exist in the usual sense, but the limit
\[ \lim_{R \to \infty} \int_{-R}^{R} \frac{P(x)}{Q(x)} \, dx \]
does exist. This limit is called the principle value of the integral.

We then have
\[ P \int_{\mathbb{R}} \frac{P(x)}{Q(x)} \, dx = \pi i \, (\text{residue at } \infty) + 2\pi i \sum \, (\text{residues in the upper half-plane}) \]

Example 1.42 (Poles on the real axis). Let $f(x)$ be analytic with a pole on the real axis at $z = x_0$. Then we have
\[ P \int_{a}^{b} f(x) \, dx = \int_{C^+} f(z) \, dz + \pi i \text{Res}(f, z_0) \]
Example 1.43 (Integrals of trigonometric functions). Let \( R(x, y) \) be a rational function bounded on the circle \(|z| = 1\). Then
\[
\int_0^{2\pi} R(\cos \theta, \sin \theta)\,d\theta = \int_{|z|=1} \frac{1}{iz} R\left(\frac{z^2 + 1}{2z}, \frac{z^2 - 1}{2iz}\right)\,dz
\]

Example 1.44. The integrals
\[
\int_R f(x)\cos \alpha x\,dx
\]
\[
\int_R f(x)\sin \alpha x\,dx
\]
are the real and imaginary parts of
\[
I = \int_R f(x)e^{i\alpha x}\,dx
\]

2. Analytic Theory of Differential Equations

2.1. Existence and uniqueness.

Theorem 2.1 (Existence and uniqueness of first order differential equations). Let \( f \) be an analytic function of two complex variables in the open polydisc \(|z - z_0| < a, |w - w_0| < b\). The first order differential equation,
\[
\frac{dw}{dz} = f(z, w)
\]
has a unique analytic solution \( w = w(z) \) such that \( w = w_0 \) when \( z = z_0 \) in some disc \(|z - z_0| < h, 0 < h \leq a\).

Proof. By repeated differentiation, we can constrict the formal Taylor series
\[
w(z) = w_0 + \sum_{n=1}^{\infty} \frac{w^{(n)}(z_0)}{n!}(z - z_0)^n
\]
in terms of \( f \) and its derivatives, which satisfies the differential equation. We can then show that this series has a positive radius of convergence.

2.2. Singular point analysis of nonlinear differential equations.

Definition 2.2 (Singular points of differential equations and their solutions). Consider the \( n^{th} \)-order differential equation,
\[
w^{(n)} = f(z, w, w', w'', \ldots, w^{(n-1)})
\]
where \( f \) is a locally analytic function of its \( n \) complex arguments. A singular point of the differential equation is a point
\[
(z_0, w_0, w'_0, \ldots, w^{(n-1)}_0) \in \mathbb{C}^n
\]
at which \( f \) is not analytic or a point where one or more of the arguments is infinite.

A fixed singularity of a differential equation is a singular hyperplane \( z = z_0 \) or \( z = \infty \).
Definition 2.3 (Classification of singularities of solutions $w(z)$). Singularities of the solutions are classified as fixed or movable. A movable singular point is a singular point of $w(z)$ depending on one or more integration constants. A fixed singular point of $w(z)$ is independent of the integration constants and is included among the fixed singularities of the differential equation.

Example 2.4.  

(i) 

$$w' = w^2$$

with solution $w = -\frac{1}{z-C}$. This has a movable pole at $z = C$.

(ii) 

$$w' = \frac{1}{w}$$

with solution $w = \pm \sqrt{2(z-C)}$. This has a moveable quadratic branch point at $z = C$.

(iii) 

$$w'' = (w')^2$$

with solution $w = -\log(z-C_1) + C_2$, movable logarithmic branch point at $z = C_1$.

2.3. Painlevé transcendents. Consider the differential equation 

$$w'' = 6w^2 + g(z)$$

Attempting a Laurent-type expansion about a movable singularity $z = z_0$ leads us to find that the movable terms balance if and only if $p = 2$, $a_0 = 1$. We then find the recurrence relation for $a_n$. We find that it is of the form

$$(n + 1)(n - 6)a_n = f_n(a_0, a_1, \ldots)$$

and so there is a possible obstruction at $n = 6$ - a resonance number. To resolve this, we introduce logarithmic terms.

This introduces a log-pole at $z_0$, a logarithmic branch point. To avoid this, we find that we must set $g''(z_0) = 0$, and hence our original differential equation is 

$$w'' = 6w^2 + \alpha z + \beta$$

called the Painlevé-I transcendent. When $\alpha = 0$, the system admits the first integral $(w')^2 = 4w^3 + 2\beta w + K$, which is solved with a Weierstrass elliptic function, having one double pole in each period parallelogram.

Consider the differential equation 

$$w'' = 2w^3 + C(z)w + D(z).$$

Similar analysis yields that $C''(z) = 0$, $D'(z) = 0$. Hence, the differential equation 

$$w'' = 2w^3 + (\alpha z + \beta)w + \gamma$$
When $\alpha = 0$, it has the first integral
\[(w')^2 = w^4 + \beta w^2 + 2\gamma w + K\]
which is solved by Jacobi elliptic functions. Each period parallelogram has two simple poles, one with residue $a_0 = 1$, the other with $a_0 = -1$. When $\alpha \neq 0$, the DE above defines a new function known as the Painlevé-II transcendent.

2.4. Fuchsian theory.

\[w^{(n)} + p_1(z)w^{(n-1)} + \cdots + p_n(z)w = R(z)\]

The fixed singularities of the solution $w(z)$ are included among the singularities of the $p_i(z)$, $i = 1, 2, \ldots, n$, and $R(z)$ and possibly $z = \infty$. Linear DE’s cannot have movable singularities.

**Theorem 2.5.** Suppose $z_0$ is not a singularity of the $p_i(z)$ or $R(z)$. Then the solution of the above linear DE satisfying initial conditions $w^{(i)}(z) = w^{(i)}(z_0)$ is analytic in a disc centred at $z_0$ with radius equal to the distance between $z_0$ and the nearest singularity of the $p_i(z)$ and $R(z)$.

**Definition 2.6** (Regular singular points). Consider the linear homogeneous DE

\[w^{(n)} + p_1(z)w^{(n-1)} + \cdots + p_n(z)w = 0\]

where the $p_i(z)$ are rational functions. The DE has a regular singular point at $t = z_1$ if

- at least one of the $p_i$ has a pole at $z = z_1$
- the order of the pole of $p_i(z)$ at $z = z_1$ is at most $i$ for all $i$.

We have $z = \infty$ is a regular singular point if

\[p_i(z) = O\left(\frac{1}{z^i}\right)\]
as $z \to \infty$ for all $i$.

Near a regular singular point at $z = z_1$, one or more particular solutions can be constructed as a power of $z - z_1$ times a convergent power series;

\[(z - z_0)^p \{a_0 + a_1(z - z_0) + \ldots \}\]
The leading powers $p$ satisfy the indicial equation

\[p(p - 1)(p - 2) \ldots (p - 2 + 1) + q_1 p(p - 1)(p - 2) \ldots (p - n + 2) + q_{n-1}p + q_n = 0\]
where $q_i = \lim_{z \to z_1} (z - z_1)^i p_i(z)$

**Definition 2.7** (Fuchsian differential equation). A Fuchsian DE is a linear homogeneous DE all of whose singular points are regular.

Hence
all the \( p_i(z) \) are rational functions,
- If \( z_1 \) is a pole of any of the \( p_i \) then the order of the pole in \( p_i \) is less than \( p_i \) for all \( i \),
- as \( z \to \infty \),

\[
p_i(z) = O\left(\frac{1}{z^i}\right)
\]

**Definition 2.8** (Möbius transformations). Fuchsian character is preserved under \( \tilde{z} = \frac{az+b}{cz+d} \) with \( ad - bc \neq 0 \). The singular points change position, but their exponents are not affected

If \( z_1 \) is a regular singular point, the transformation

\[
\tilde{w} = (z - z_1)^\lambda w
\]

has the following effect:

- all the exponents at \( z_1 \) are raised by \( \lambda \),
- all the exponents at \( \infty \) are lowered by \( \lambda \),
- exponents are other points are not affected.

**Theorem 2.9** (Quadratic transformation). The transformation \( z = z^2 \) has the following effect:

- Exponents at \( z = 0 \) and \( z = \infty \) are doubled.
- A singular point at \( z + 1 \neq - \) splits into a pair of singular points at \( \pm \sqrt{z_1} \) with the **same** exponents

2.5. **Hypergeometric functions.** The hypergeometric differential equation

\[
z(1 - z)w'' + (\gamma - (1 + \alpha + \beta)z)w' - \alpha \beta w = 0
\]

is a Fuchsian DE with regular points at 0, 1 and \( \infty \), with exponents

- \( z = 0 \) exponents \( 0, 1 - \gamma \),
- \( z = 1 \) exponents \( 0, \gamma - \alpha - \beta \),
- \( z = \infty \) exponents \( \alpha, \beta \)

The **hypergeometric function** \( F(\alpha, \beta; \gamma; z) \) is the particular solution

\[
F(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n
\]

where \( (\alpha)_n = \alpha(\alpha + 1) \ldots (\alpha + n - 1) = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} \).

The general solution of the hypergeometric equation is

\[
y = C_1 F(\alpha, \beta; \gamma; z) + C_2 z^{1-\gamma} F(1 + \alpha - \gamma, 1 + \beta - \gamma; 2 - \gamma; z)
\]

We have an integral formula

\[
F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \beta) \Gamma(\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-1}(1-zt)^{-\alpha} dt
\]
valid for $\Re \gamma > \Re \beta > 0$, $|z| < 1$. but can be extended to $z \in \mathbb{C} = [1, \infty)$.

2.6. Gauss’ formula at $z = 1$.

$$F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}$$

2.7. Kummer’s formula at $z = -1$.

$$F(\alpha, \beta; 1 + \beta - \alpha; -1) = \frac{1}{2} \frac{\Gamma(\frac{1}{2}\beta)\Gamma(1 + \beta - \alpha)}{\Gamma(\beta)\Gamma(1 + \frac{1}{2}\beta - \alpha)}$$

2.8. Evaluation at $z = \frac{1}{2}$.

$$F(\alpha, \beta; \gamma; \frac{1}{2}) = 2^{\gamma} F(\alpha, \gamma - \beta; \gamma; -1)$$

$$F(\alpha, \beta; \gamma; \frac{1}{2}) = \frac{\Gamma(\frac{1}{2}\gamma)\Gamma(\frac{1}{2}\gamma + \frac{1}{2})}{\Gamma(\frac{1}{2}\alpha + \frac{1}{2}\gamma)\Gamma(\frac{1}{2} - \frac{1}{2}\alpha + \frac{1}{2}\gamma)}$$

2.9. Connection formulae. We have

$$F(\alpha, \beta; 1 + \alpha + \beta - \gamma; 1 - z) =$$

$$AF(\alpha, \beta; \gamma; z) +$$

$$B z^{1-\gamma} F(1 + \alpha - \gamma, 1 + \beta - \gamma; 2 - \gamma; z)$$

where

$$A = \frac{\Gamma(1 - \gamma)\Gamma(1 + \alpha + \beta - \gamma)}{\Gamma(1 + \alpha - \gamma)\Gamma(1 + \beta - \gamma)}$$

$$B = \frac{\Gamma(\gamma - 1)\Gamma(1 + \alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$F(\alpha, \beta; \gamma; z) =$$

$$A_1 F(\alpha, \beta; 1 + \alpha + \beta - \gamma; 1 - z) +$$

$$B_1 (1 - z)^{\gamma - a - \beta} F(\gamma - a, \gamma - \beta - 1 - a - \beta + \gamma; 1 - z)$$

where

$$A_1 = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}$$

$$B_1 = \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)}$$

There are various similar formula on page 44 of the notes.

2.10. Elliptic functions. There are three main approaches to the study of elliptic functions:
2.11. **Elliptic integrals.** Consider the class of indefinite integrals,

\[ \int R(x, \sqrt{P(x)}) \, dx, \]

where \( R \) is a rational function of two variables and \( P \) is a polynomial without square factors. When \( P(x) \) has degree 3 or 4, a Möbius transform of the polynomial \( P(x) \) can be given one of several equivalent normalisations:

**Jacobi** \( P(x) = (1 - x^2(1 - k^2x^2)) \),

**Weierstrass** \( P(x) = 4x^3 - g_2x - g_3 \)

**Elliptic integral of the first kind:**

\[ F(k, \phi) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \]

**Elliptic integral of the second kind:**

\[ E(k, \phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \theta} \, d\theta \]

**Elliptic integral of the third kind:**

\[ \Pi(n, k\phi) = \int_0^\phi \frac{d\theta}{(1 + n^2 \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}} \]

**Complete elliptic integral of the first kind:**

\[ K(k) = F(k, \frac{\pi}{2}) \]

\[ = \int_0^1 \frac{dx}{\sqrt{1 - x^2}\sqrt{1 - k^2x^2}} \]

\[ = \frac{1}{2} \pi F\left(\frac{1}{2}, \frac{1}{2}, 1; k^2\right) \]

**Complete elliptic integral of the second kind:**

\[ E(k) = E(k, \frac{\pi}{2}) \]

\[ = \int_0^1 \frac{\sqrt{1 - k^2x^2}}{\sqrt{1 - x^2}} \, dx \]

\[ = \frac{1}{2} \pi F\left(-\frac{1}{2}, \frac{1}{2}, 1; k^2\right) \]
2.12. Inversion of elliptic integrals. The Jacobi elliptic function \( \text{sn} \, z \) is defined by

\[
\int_0^{\text{sn} \, z} \frac{dt}{\sqrt{1-t^2} \sqrt{1-k^2t^2}} = z
\]

or equivalently, by the differential equation

\[
(w')^2 = (1 - w^2)(1 - k^2w^2)
\]

with \( w(0) = 0 \) and \( w'(0) > 0 \).

Then we define \( \text{cn} \, z, \text{dn} \, z \) by

\[
\text{cn} \, z = \sqrt{1 - \text{sn}^2 z}
\]

\[
\text{dn} \, z = \sqrt{1 - k^2 \text{sn}^2 z}
\]

2.13. Doubly periodic meromorphic functions. We define

\[
\wp(z) = \frac{1}{z^2} + \sum_{m,n \in \mathbb{Z}} \left( \frac{1}{(z - mw - nw')^2} - \frac{1}{(mw + nw')^2} \right)
\]

where \( w \) and \( w' \) are nonzero complex numbers with \( w'/w \) not real. It is easy to see that \( \wp(z) \) is globally meromorphic with periods \( w \) and \( w' \).

If \( f \) is any elliptic function, then \( \int_C f(z) = 0 \), where \( C \) is the period parallelogram, as the integrals on opposite sides cancel. Hence, the sum of the residues of all the poles in a period parallelogram.

We have

\[
\wp'(z) = -2 \sum_{m,n \in \mathbb{Z}} \frac{1}{(z - mw - nw')^3}
\]

Consider the Laurent expansion of \( \wp \) and \( \wp' \) about \( z = 0 \). We have

\[
\frac{1}{(z - mw - nw')^2} = \sum_{k=0}^{\infty} \frac{(k+1)z^k}{(mw - nw')k+2}
\]

Defining \( I_{2k} = \sum' \frac{1}{(mw+nw')^{2k}} \) gives

\[
\wp(z) = \frac{1}{z^2} + 3I_4 z^2 + 5I_6 z^4 + \ldots
\]

\[
\wp'(z) = -\frac{2}{z^3} + 6I_4 z + 20I_6 z^3
\]

where the radius of convergence is the minimum of \( |w| \) and \( |w'| \).

Letting \( g_2 = 60I_4, g_3 = 140I_6 \), we obtain the DE

\[
(w')^2 = 4w^3 - g_2w - g_3
\]

with general solution \( \wp(z - z_0) \).
We also have
\[ \int_{\gamma} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}} = z \]

2.14. Jacobi elliptic functions. We have
\[
\begin{align*}
\text{sn}'z &= \text{cn}z \text{dn}z \\
\text{cn}'z &= -\text{sn}z \text{dn}z \\
\text{dn}'z &= -k^2 \text{sn}z \text{cn}z
\end{align*}
\]
deduced from \( \text{sn}z \) satisfying \( w' = \sqrt{1-w^2} \sqrt{1-k^2w^2} \).

For identities of the Jacobi elliptic functions, see pages 57-67 in the notes.

2.15. Addition theorems. See pages 57-67 in the notes.

2.16. Liouville theory. Let an elliptic function be defined as any doubly periodic meromorphic function.

Definition 2.10 (Order). The order of an elliptic function is the number of poles of \( f(z) \) inside a period parallelogram, counting multiplicities (a pole of order \( n \) is counted as \( n \) poles.) Then \( \varphi, \text{sn}, \text{cn} \) are elliptic functions of order 2.

Theorem 2.11. An elliptic function of order zero is constant.

Theorem 2.12. The sum of the residues of \( f(z) \) at all poles in a period parallelogram is zero.

Theorem 2.13. The transcendental equation \( f(z) = a \) where \( f(z) \) is an elliptic function of order \( m \) has exactly \( m \) roots in every period parallelogram, counting multiplicities, for every \( a \in \mathbb{C} \).

Theorem 2.14. A Möbius transformation leaves the order and periods of an elliptic function unchanged.

Theorem 2.15. Any solution of \( (w')^2 = aw^4 + bw^3 + cw^2 + dw + e \) with \( a, b \) both not zero, no square factors, is an elliptic function of order 2.

2.17. Weierstrass’ Theorem. Let \( f(z) \) be an elliptic function of any order. Then it has a unique representation
\[ f(z) = R_1(\varphi(z)) + R_2(\varphi(z))\varphi'(z) \]
where \( \varphi(z) \) is the Weierstrass function having the same periods and \( R_i \) rational functions.