

# AMH4 - ADVANCED OPTION PRICING

ANDREW TULLOCH

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## 1. THEORY OF OPTION PRICING

**Definition 1.1** (Brownian motion). A process  $W_t$  is a  $\mathbb{P}$ -Brownian motion if it satisfies

- (1)  $W_t$  is continuous with  $W_0 = 0$  (a.s.)
- (2)  $W_t$  has stationary and independent increments.
- (3) For any  $t > 0$ ,  $W_t \sim N(0, t)$  under the probability measure  $\mathbb{P}$ .

**Theorem 1.2** (Properties of conditional expectation). Assume we have a probability space  $(\Omega, \mathbb{P})$  and  $\sigma$ -algebras  $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2$ . Assume that  $\mathcal{G}_2 \subset \mathcal{G}_1$ . Then

- (1) If  $X$  is a random variable, then

$$\mathbb{E}(X | \mathcal{G}_2) = \mathbb{E}(\mathbb{E}(X | \mathcal{G}_1) | \mathcal{G}_2)$$

- (2) If  $Y$  is a  $\mathcal{G}$ -measurable random variable, then

$$\mathbb{E}(XY | \mathcal{G}) = Y\mathbb{E}(X | \mathcal{G})$$

**Definition 1.3** (Martingale). A stochastic process  $X_t$  is a  $\mathcal{F}_t$ -martingale if  $\mathbb{E}(|X_t|) < \infty$  and

$$X_s = \mathbb{E}(X_t | \mathcal{F}_s)$$

for all  $s \leq t$ .

**Theorem 1.4** (Itô's lemma). If  $F(X_t, t)$  is  $C_{2,1}$  and  $dX_t = \alpha_t dt + \beta_t dW_t$ , then

$$dF = (F_t + \alpha F_x + \frac{1}{2}\beta^2 F_{xx}) dt + \beta F_x dW_t$$

**Lemma 1.5** (Product and Quotient rule). Let  $X_t$  be an Itô processes, so that

$$dX_t = \alpha dt + \beta dW_t.$$

Let  $F(X_t, t), G(X_t, t)$  be  $C_{2,1}$ . Then

$$\begin{aligned} d(FG) &= (F dG + G dF) + \beta^2 F_x G_x dt \\ d(F/G) &= \frac{G dF - F dG}{G^2} + \frac{\beta^2 G_x}{G^3} (FG_x - GF_x) dt \end{aligned}$$

**Lemma 1.6** (Itô isometry). If  $\sigma_s \in L^2$ , then

$$\mathbb{E}\left(\int_0^t \sigma_s dW_s\right)^2 = \mathbb{E}\left(\int_0^t \sigma_s^2 ds\right)$$

**Definition 1.7** (Local martingale).  $X_t$  is a local martingale if there exists a sequence of stopping times  $\nu_n$  such that for every  $n$ , the process  $X_t^n = X_{\min(\nu_n, t)}$  is a martingale.

**Theorem 1.8** (Martingale representation theorem). Let  $\mathcal{F}_t$  be the natural filtration of a Brownian motion.

(1) Any progressively measurable process  $\sigma_t$  satisfying

$$\mathbb{P}\left(\int_0^t \sigma_s^2 ds\right) < \infty = 1 \quad \forall t$$

the process

$$t \mapsto \int_0^t \sigma_s dW_s$$

is a local martingale.

(2) If  $X_t$  is an  $L^2$  martingale, then there exists a progressively measurable process  $\sigma_s$  such that

$$X_t = \int_0^t \sigma_s dW_s$$

Hence the Brownian martingales (martingales with respect to the Brownian filtration) are essentially the Itô integrals.

**Theorem 1.9** (Girsanov). Let  $\lambda_t$  be progressively measurable with

$$\mathbb{E} \exp\left(\frac{1}{2} \int_0^T \lambda^2(t) dt\right) < \infty$$

Then there exists a measure  $\mathbb{P}^*$  such that

(1)  $\mathbb{P}^*$  is equivalent to  $\mathbb{P}$ ,

(2)

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(-\int_0^t \lambda_s dW_s - \frac{1}{2} \int_0^t \lambda_s^2 ds\right)$$

(3)  $W_t^* = W_t + \int_0^t \lambda_s ds$  is a  $\mathbb{P}^*$ -Brownian motion

As a partial corollary, if  $\mathbb{P}^*$  is equivalent to  $\mathbb{P}$  then there exists a progressively measurable process  $\lambda_t$  such that

$$W_t^* = W_t + \int_0^t \lambda_s ds$$

is a Brownian motion under  $\mathbb{P}^*$ .

**Corollary.** We can then use Girsanov's theorem to transform a Brownian motion with drift to a martingale. e.g. Under  $\mathbb{P}$ ,

$$\begin{aligned} dX_t &= \mu_t dt + \sigma_t dW_t \\ &= \sigma_t d\left(W_t + \int_0^t \sigma_s^{-1} \mu_s ds\right) \\ &= \sigma_t dW_t^* \end{aligned}$$

where we set  $\lambda_s = \sigma_s^{-1} \mu_s$  in Girsanov's theorem.

**Theorem 1.10** (Multivariate Itô's lemma). *Let  $dX_{i,t} = \alpha_i dt + \beta_i dW_{i,t}$  with  $W_{i,t}$  correlated Brownian motions. Then if  $F(X_{1,t}, \dots, X_{n,t}, t)$  is  $C_{2,1}$ , then*

$$dF = \left( F_t + \sum_{i=1}^n \alpha_i F_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \beta_i \beta_j \rho_{ij} F_{ij} \right) dt + \sum_{i=1}^n \beta_i F_i dW_i(t)$$

## 2. BLACK-SCHOLES PDE METHOD

**Theorem 2.1** (Black-Scholes PDE). *Let  $f(X_t, t)$  represent the price of a contingent claim on an asset  $X_t$ , where  $X_t$  is assumed to follow geometric Brownian motion. Under certain assumptions, we can derive the Black-Scholes PDE,*

$$f_t = rf - rxf_x - \frac{1}{2}\sigma^2 x^2 f_{xx}$$

Solving the Black-Scholes PDE along with initial conditions and payoff at expiration yields the function  $f(X_t, t)$  which gives the option value at any time  $t$  and any underlying value  $X_t$ .

## 3. MARTINGALE METHOD

Consider a market with risky security  $X_t$  and riskless security  $B_t$ .

**Definition 3.1** (Contingent claim). A random variable  $C_T : \Omega \rightarrow \mathbb{R}$ ,  $\mathcal{F}_T$ -measurable is called a contingent claim. If  $C_T$  is  $\sigma(X_T)$ -measurable it is **path-independent**.

**Definition 3.2** (Strategy). Let  $\alpha_t$  represent number of units of  $X_t$ , and  $\beta_t$  represent number of units of  $B_t$ . If  $\alpha_t, \beta_t$  are  $\mathcal{F}_t$ -adapted, then they are strategies in our market model. Our strategy value  $V_t$  at time  $t$  is

$$V_t = \alpha_t X_t + \beta_t B_t$$

**Definition 3.3** (Self-financing strategy). A strategy  $(\alpha_t, \beta_t)$  is self financing if

$$dV_t = \alpha_t dX_t + \beta_t dB_t$$

The intuition is that we make one investment at  $t = 0$ , and after that only rebalance between  $X_t$  and  $B_t$ .

**Definition 3.4** (Admissible strategy).  $(\alpha_t, \beta_t)$  is an **admissible strategy** if it is self financing and  $V_t \geq 0$  for all  $0 \leq t \leq T$ .

**Definition 3.5** (Arbitrage). An arbitrage is an admissible strategy such that  $V_0 = 0, V_T \geq 0$  and  $\mathbb{P}(V_T > 0) > 0$ .

**Definition 3.6** (Attainable claim). A contingent claim  $C_T$  is said to be attainable if there exists an admissible strategy  $(\alpha_t, \beta_t)$  such that  $V_T = C_T$ . In this case, the portfolio is said to replicate the claim. By the law of one price,  $C_t = V_t$  at all  $t$ .

**Definition 3.7** (Complete). The market is said to be **complete** if every contingent claim is attainable

**Theorem 3.8** (Harrison and Pliska). Let  $\mathbb{P}$  denote the real world measure of the underlying asset price  $X_t$ . If the market is arbitrage free, there exists an equivalent measure  $\mathbb{P}^*$ , such that the discounted asset price  $\hat{X}_t$  and every discounted attainable claim  $\hat{C}_t$  are  $\mathbb{P}^*$ -martingales. Further, if the market is complete, then  $\mathbb{P}^*$  is unique. In mathematical terms,

$$C_t = B_t \mathbb{E}^*(B_T^{-1} C_T | \mathcal{F}_t).$$

$\mathbb{P}^*$  is called the equivalent martingale measure (EMM) or the risk-neutral measure.

#### 4. MONTE CARLO METHODS

**4.1. Method of antithetic variances.** Instead of simulating  $X$ , also simulate a random variable  $Z$  with the same variance and expectation as  $X$ , but is negatively correlated with  $X$ . Then take as  $Y$  the random variable

$$Y = \frac{X + Z}{2}$$

Obviously  $\mathbb{E}(Y) = \mathbb{E}(X)$ . On the other side, we have

$$\begin{aligned} \text{Var}(Y) &= \text{Cov}\left(\frac{X + Z}{2}, \frac{X + Z}{2}\right) \\ &= \frac{1}{4} \text{Var}(X) + 2\text{Cov}(X, Z) + \text{Var}(Z) \leq \frac{1}{2} \text{Var}(X) \end{aligned}$$

So we can reduce variance by a factor of two.

#### 4.2. Control variate method.

**Theorem 4.1.** Suppose we seek to estimate  $\theta = \mathbb{E}(Y)$  where  $Y = h(X)$  is the outcome of a simulation. Suppose that  $Z$  is also an output of the simulation, and assume that  $\mathbb{E}(Z)$  is known.

Let

$$c = \frac{\text{Cov}(Y, Z)}{\text{Var}(Z)}. \quad (\ddagger)$$

Then

$$\hat{\theta}_c = Y + c(\mathbb{E}(Z) - Z) \quad (\dagger)$$

is an unbiased estimator of  $\theta$ , and if  $\text{Cov}(Y, Z) \neq 0$ ,  $\hat{\theta}_c$  has a lower variance than  $\hat{\theta} = Y$ , and indeed has the lowest variance for all estimators of the form

$$\hat{\theta}_\gamma = Y + \gamma(\mathbb{E}(Z) - Z)$$

*Proof.* We have

$$\text{Var}(\hat{\theta}_c) = \text{Var}(Y) + c^2 \text{Var}(Z) - 2c \text{Cov}(Y, Z). \quad (\star)$$

From elementary methods of calculus, we see that  $\text{Var}\hat{\theta}_c$  is minimised at

$$c = \frac{\text{Cov}(Y, Z)}{\text{Var}(Z)}$$

Substituting in this value for  $c$  in  $(\star)$ , we obtain

$$\begin{aligned} \text{Var}(\hat{\theta}_c) &= \text{Var}(Y) - \frac{\text{Cov}(Y, Z)^2}{\text{Var}(Z)} \\ &= \text{Var}(\hat{\theta}) - \frac{\text{Cov}(Y, Z)^2}{\text{Var}(Z)} \end{aligned}$$

and thus we only need  $\text{Cov}(Y, Z) \neq 0$  to obtain our variance reduction.

In practice, we do not know  $\text{Cov}(Y, Z)$ . Thus, we have to do a number of *burn-in* simulations to generate  $Y$  and  $Z$ , and then compute an estimate  $\hat{c}$  to use in the full simulation.  $\square$

## 5. NUMERICAL SIMULATION OF STOCHASTIC DIFFERENTIAL EQUATIONS

**Theorem 5.1.** *Let*

$$dX_t = a(t, X_t) dt + b(t, X_t) dB_t$$

*Assume  $\mathbb{E}X_0 < \infty$ .  $X_0$  is independent of  $B_s$  and there exists a constant  $c > 0$  such that*

- (1)  $|a(t, x)| + |b(t, x)| \leq C(1 + |x|)$ .
- (2)  $a(t, x), b(t, x)$  satisfy the Lipschitz condition in  $x$ , i.e.

$$|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leq C|x - y|$$

*for all  $t \in (0, T)$ .*

*Then there exists a **unique (strong) solution**.*

**Definition 5.2** (Strong convergence). A numerical scheme for solving an SE is said to converge with strong order  $\gamma$ , if for sufficiently small  $\Delta$ , we have

$$\mathbb{E}(|X(T) - X_N|) \leq K_T \Delta^\gamma$$

This implies that the generated paths approximate the true paths of the SDE - and so one calls this path-wise convergence or strong convergence.

**Definition 5.3** (Weak convergence). A numerical scheme for solving an SDE is said to converge with weak order  $\beta$  if for sufficiently small  $\Delta$  and each polynomial  $g$ , we have

$$|\mathbb{E}(g(X_T)) - \mathbb{E}(g(X_N))| \leq K_{g,T} \Delta^\beta$$

Note that strong convergence always implies weak convergence.

Note also that strong convergence implies pathwise convergence. This is true by Markov's inequality, we have

$$\begin{aligned}\mathbb{P}(|X_n - X(T)| \geq \Delta^{\beta/2}) &\leq \frac{\mathbb{E}(|X_n - X(T)|)}{\Delta^{\beta/2}} \\ &\leq C \frac{\Delta^\beta}{\Delta^{\beta/2}}\end{aligned}$$

- Note.*
- (1) Weak convergence is basically convergence in distribution, but it has no path-wise properties.
  - (2) If terms like  $\mathbb{E}(h(X_T))$  are computed via Monte Carlo, then the weak convergence concept is sufficient.
  - (3) If the option is a path dependent option, then strong convergence is the right concept, as the payoff depends on the whole path, rather than the distribution of the terminal value of the stock.

**Theorem 5.4** (Euler-Maruyama scheme).

$$\begin{aligned}X_0 &= X(0) \\ X_{n+1} &= X_n + a(t_n, X_n)\Delta t_n + b(t_n, X_n)\Delta W_n\end{aligned}$$

where

$$\begin{aligned}\Delta t_n &= t_{n+1} - t_n \\ \Delta W_n &= W_{t_{n+1}} - W_{t_n}\end{aligned}$$

*Euler-Maruyama has **strong convergence order**  $\gamma = \frac{1}{2}$  and **weak convergence order**  $\beta = 1$ .*

**Theorem 5.5** (Milstein scheme). *Consider the homogenous scalar stochastic differential equation*

$$dX_t = a(X_t) dt + b(X_t) dW_t$$

$$\begin{aligned}X_0 &= X(0) \\ X_{n+1} &= X_n + a(X_n)\Delta t_n + b(X_n)\Delta W_n + \frac{1}{2}b'(X_n)b(X_n)((\Delta W_n)^2 - \Delta t_n)\end{aligned}$$

*One can prove that the Milsten scheme has **strong and weak convergence order**  $\gamma = 1$ .*

## 6. STOCHASTIC OPTIMAL CONTROL

**Definition 6.1** (Controlled stochastic differential equation).

$$dx(t) = f(t, x(t), u(t)) dt + \sigma(t, x(t), u(t)) dW(t)$$

where  $u(t, \omega) = u(t, x(t, \omega))$  is a stochastic process, known as the **control**.

**Definition 6.2** (Admissible control). A control  $u$  is called admissible for the constraints if for every initial value  $x_0 \in S$  the corresponding stochastic differential equation has a unique solution with  $x(0) = x_0$  and  $u(t, \omega) \in \mathcal{U}$  for all  $t \in [0, \infty]$ . We denote the set of admissible controls with  $\mathcal{A}$ .

**Definition 6.3** (Stochastic optimal control problem). We seek to solve

$$\max_{u \in \mathcal{A}} \mathbb{E} \left[ \int_0^T e^{-rt} B(t, x(t), u(t)) dt + e^{-rT} S(x(T)) \cdot \mathbf{1}_{T < \infty} dt \right]$$

under the dynamic constraint

$$dx(t) = f(t, x(t), u(t)) dt + \sigma(t, x(t), u(t)) dW(t)$$

with initial condition  $x(0) = x_0$ , and discount rate  $r > 0$ .

$B$  is called the benefit function,  $S$  is called the final payoff, and the control  $u$  is called the optimal control, and the optimal value is called the value of the problem.

**Definition 6.4** (Value function).

$$\max_{u \in \mathcal{A}} \mathbb{E} \left[ \int_t^T e^{-r(s-t)} B(s, x(s), u(s)) ds + e^{-r(T-t)} S(x(T)) \cdot \mathbf{1}_{T < \infty} dt \mid x(t) = x \right]$$

subject to

$$\begin{aligned} dx(s) &= f(s, x(s), u(s)) ds + \sigma(s, x(s), u(s)) dW(s) \\ x(t) &= x \end{aligned}$$

Note that  $V(0, x_0)$  is the value of the optimal control problem.  $V(t, x)$  is the value of the problem, if we started at time  $t$  with initial state  $x$ .

**Theorem 6.5** (Hamilton-Jacobi-Bellman equation). Assume  $T < \infty$ . Let  $V : [0, T] \times S \rightarrow R$  be a  $C_{1,2}$  function and assume it satisfies the HJB equation

$$\begin{aligned} rV(t, x) - V_t(t, x) &= \max_{u \in \mathcal{A}} \left( B(t, x, u) + V_x(t, x) f(t, x, u(t)) + \frac{1}{2} \text{tr}(V_{xx}(t, x) \sigma(t, x, u) \sigma(t, x, u)^T) \right) \\ V(T, x) &= S(x). \end{aligned}$$

Let  $\phi(t, x)$  be the set of maximisers of the right hand side and let  $u^* \in \mathcal{A}$  such that  $u^*(t, \omega) \in \phi(t, x(t, \omega))$  for all  $t \in [0, T], \omega \in \Omega$ . Then  $u^*$  is the optimal control and  $V$  is the value function for the stochastic optimal control problem.

**Theorem 6.6** (Hamilton-Jacobi-Bellman equation, infinite time). Consider the time homogenous, infinite time horizon problem

$$\max_{u \in \mathcal{A}} \mathbb{E} \left[ \int_0^\infty e^{-rt} B(x(t), u(t)) dt \right]$$



subject to

$$dx(t) = f(x(t), u(t)) dt + \sigma(x(t), u(t)) dW_t.$$

Then the value function is independent of  $t$ , and so  $V(t, x) = V(x)$ , and the optimal control is of the type  $u(t, x) = u(x)$ . The HBJ equation in this case becomes the ODE

$$rV(x) = \max_u \left( B(x, u) + V'(x)f(x, u) + \frac{1}{2}V''(x)\sigma(x, u)^2 \right)$$